# Total score (100 points)：

**1** *(10 points) Find the general solution of* $x' = Ax$ *with*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}.$$

*Here, eigenvalues of the matrix* $A$ *are* $1$, $1$, *and* $0$.

**2** *(a) (5 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

*is equivalent to the system*

$$\frac{dx}{dt} = y$$
$$\frac{dt}{dt} = -x - \mu(x^2 - 1)y$$

*(b) (15 points) Find the stabilities of the critical point* $(0, 0)$ *for the cases* $\mu > 0$ *and* $\mu < 0$.

**3** *(15 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

*where* $a, b, c > 0$. *Let* $y = x'$. *Show that* $(0, 0)$ *is Liapunov stable using Liapunov function of the form* $V(x, y) = \alpha x^2 + \beta x^4 + \gamma y^2$ *for* $\alpha, \beta, \gamma > 0$.

**4** *(15 points) Consider the following system*

$$\frac{dx}{dt} = x - y - x^3$$
$$\frac{dt}{dt} = x + y - y^3$$

*Show that there is at least one stable limit cycle in the region* $A = \{(x, y) \in \mathbb{R}^2 | 1 \leq |(x, y)| \leq \sqrt{2}\}$.

**5** *(15 points) If* $C \geq 0$ *and* $u, v : [0, \beta] \to [0, \infty)$ *are continuous and*

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

*for all* $t \in [0, \beta]$, *then*

$$u(t) \leq Ce^{v(t)},$$

*where* $v(t) = \int_0^t v(s)ds$.

**6** *(a) (15 points) Let* $n = 2$. *For any* $2 \times 2$ *constant real matrix* $A$, *show that there exists an invertible real matrix* $P$ *such that the matrix*

$$B = P^{-1}AP$$

*has one of the following forms*

$$\text{(i)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ (ii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ (iii)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

*where* $\lambda, \mu, a, b \in \mathbb{R}$. *Find* $P$ *explicitly.*

*(b) (10 points) Let* $A = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$. *where* $\lambda, \mu, \alpha \in \mathbb{R}$. *Solve the initial value problem:* $x' = Ax,\ x(0) = x_0$.

Total score (100 points) ：

**1** *(20 points) Find the fundamental matrix of $x' = Ax$ with*

$$A = \begin{pmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{pmatrix}.$$

**2** *(a) (4 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$$

*is equivalent to the system*

$$\frac{dx}{dt} = y$$
$$\frac{dt}{dt} = -x - \mu(x^2-1)y$$

*(b) (16 points) Characterize the types and the stabilities of the critical point $(0,0)$ for the cases $\mu > 0$ and $\mu < 0$.*

**3** *(20 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

*where $a, b, c > 0$. Let $y = x'$. Show that $(0,0)$ is Liapunov stable using Liapunov function of the form $V(x,y) = \alpha x^2 + \beta x^4 + \gamma y^2$ for $\alpha, \beta, \gamma > 0$.*

**4** *Consider the following initial value problem (IVP)*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \tag{2}$$

*(i)(6 points) Show that the solution of the IVP(1)(2) stays in the region $D = \{(x,y) | \frac{1}{2} \le \sqrt{x^2+y^2} \le 2\}$ whenever it exists.*

*(ii)(6 points) Show that there exists a unique solution for IVP(1)(2), which exists for all $t \in \mathbb{R}$.*

*(iii)(8 points) Find the equilibrium of (1). Discuss the asymptotical behavior of the solution for IVP(1)(2) as $t \to \infty$.*

**5** *(20 points) Let* $r, k,$ *and* $f$ *be real and continuous functions which satisfy* $r(t) \geq 0$, $k(t) \geq 0$, *and*

$$r(t) \leq f(t) + \int_a^t k(s)r(s)ds, \quad a \leq t \leq b.$$

*Show that*

$$r(t) \leq f(t) + \int_a^t f(s)k(s)\exp\left[\int_s^t k(u)du\right]ds, \quad a \leq t \leq b.$$