100 學年度下學期數學系博士班資格考試
(實變分析)

本試題卷共 2 頁，計 10 題計算證明題，每題 10 分，合計 100 分。

1. Prove the *Carathéodory theorem*: A set E is measurable if and only if for every set A,

$$|A|_e = |A \cap E|_e + |A \setminus E|_e.$$

   (Note: $|A|_e|$ denotes the outer measure of A.)

2. Prove that the set of points at which a sequence of measuable real-valued functions converges (to a finite limit) is measurable.

3. Let f be a function which is upper semi-continuous and finite on a compact set E. Show that if f is bounded above on E. Show also that f assumes its maximum on E, that is, that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

4. Let $f \in L(0,1)$. Show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,\mathrm{d}x \to 0$ as $k \to \infty$.

5. Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y \mid (x,y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero, and the for almost every $y \in \mathbb{R}^1$, $\{x \mid (x,y) \in E\}$ has measure zero.

6. (a) Write out the definition of the essential supremum $\|f\|_\infty$ of a real-valued measurable function f on a measurable set E.

   (b) Let f be a real-valued measurable function on $[0,1]$. Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

7. Let E be a measurable set in $\mathbb{R}^n$, and $0 < p < q \leq \infty$.

   (a) Prove that $L^p(E) \cap L^\infty(E) \subset L^q(E)$.

   (b) Prove that if $|E| < \infty$, then $L^q(E) \subset L^p(E)$.

8. Let $f, g \in L^2(\mathbb{R}^n)$. Prove that $f + g \in L^2(\mathbb{R}^n)$ and $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

(背面尚有試題)

9. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0,1]$, and $\{c_k\}$ be the Fourier series of a function $f \in L^2[0,1]$ with respect to the system $\{\varphi_k\}$.

   (a) Prove that the Bessel's inequality $\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} \leq \|f\|_2$ holds.

   (b) Find a necessary and sufficient condition so that the Parseval's identity $\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} = \|f\|_2$ holds, and prove your answer.

10. Let $C[0,1]$ denote the set of all real-valued continuous functions on $[0,1]$, and the linear operator $T : C[0,1] \to \mathbb{R}$ be defined by $T(f) = f(1)$ for all $f \in C[0,1]$.

    (a) Prove that T is a continuous linear functional on $C[0,1]$.

    (b) Prove that there exists an extension $T^* : L^\infty[0,1] \to \mathbb{R}^n$ of T such that T^* is a continuous linear functional on $L^\infty[0,1]$, but there is no $g \in L^1[0,1]$ satisfying

$$T^*(f) = \int_{[0,1]} (f \times g)\,\mathrm{d}x \qquad \text{for all } f \in C[0,1].$$

(試題結束)

101 學年度上學期數學系博士班資格考試
(實變分析)

本試題卷共 2 頁，計 10 題計算證明題，每題 10 分，合計 100 分。

1. Let E be a measurable subset of $\mathbb{R}$, with $|E| > 0$. Prove that there exists a positive real number ε such that $(-\varepsilon, \varepsilon) \subset E - E$, where
$$E - E = \{x - y \mid x, y \in E\}.$$

2. Prove or disprove:
   (a) Any function $f : [a,b] \to \mathbb{R}$ of bounded variation is measurable.
   (b) Any upper semicontinuous function $f : [a,b] \to \mathbb{R}$ is measurable.

3. Let E be a measurable set in $\mathbb{R}^n$ of finite measure. Prove that $f : E \to \mathbb{R}$ is measurable if and only if for any $\varepsilon > 0$, there exists a closed subset F of E such that $|E \setminus F| < \varepsilon$, and f is continuous on F.

4. (a) State without proof the Egorov's theorem.
   (b) Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E with $|E| < \infty$. If f_k converges to f a.e. in E, and $\sup_k |f_k - f| \in L(E)$, prove that $\lim_{k\to\infty} \int_E f_k = \int_E f$.

5. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfy for each $x \in [0,1]$, $f(x,y)$ is a Lebesgue integrable function of y, and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Prove that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each $x \in [0,1]$, and
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{[0,1]} f(x,y)\,\mathrm{d}y = \int_{[0,1]} \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y.$$

6. (a) State the definition for a finite function f on a finite interval $[a,b]$ to be *absolutely continuous*.
   (b) Show that the function $f(x) = x^{\alpha}$ is absolutely continuous on every bounded subinterval of $[0,\infty)$ whenever $\alpha > 0$.

7. Let $a_1, a_2, \ldots, a_N$ be non-negative real numbers, $p_1, p_2, \ldots, p_N$ be positive real numbers with $\sum_{j=1}^{N}(1/p_j) = 1$. Show that
$$\prod_{j=1}^{N} a_j \leq \sum_{j=1}^{N} \frac{a_j}{p_j}.$$

(背面尚有試題)

8. Let ℓ^∞ denote the normed linear space of all bounded real sequences. Is ℓ^∞ separable? Justify your answer.

9. Suppose that $f_k, f \in L^2$, and that $\int f_k g \to \int f g$ for all $g \in L^2$. If $\|f_k\|_2 \to \|f\|_2$, show that $f_k \to f$ in L^2 norm.

10. Let Σ be a σ-algebra on a set $\mathscr{S}$, $\{E_k\}$ be any sequence of sets in Σ, and ϕ be a non-negative additive set function on Σ. Prove that

$$\phi\left(\liminf_{k\to\infty} E_k\right) \le \liminf_{k\to\infty} \phi(E_k).$$

(試題結束)

103 學年度數學系博士班資格考試

（實 變 分 析）

※ 本試題卷共 8 題證明題

1. (a) Prove that every Borel measurable subset in $\mathbb{R}^n$ is Lebesgue measurable.

(b) Prove that there is a Lebesgue measurable subset in $\mathbb{R}^n$ is not Borel measurable.

(10%)

2. Prove or disprove (Please explain your answer):

(a) If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then f is Lebesgue measurable.

(b) If E is a Lebesgue measurable subset of $\mathbb{R}$, with $|E|>0$, then there exist $x,\ y\in E$ with $x\neq y$ such that $x-y$ is a rational number.

(c) If for each rational number a, the set $\{x\in\mathbb{R}^n \mid f(x)>a\}$ is Lebesgue measurable, then $f:\mathbb{R}^n\to\mathbb{R}$ is Lebesgue measurable.

(d) There exists a Riemann integrable function $f:[0,1]\to[0,1]$ such that f is continuous at each rational point and discontinuous at each irrational point of $[0,1]$.

(e) If f is Lebesgue integrable over E, then f is finite a.e. in E. (30%)

3. Prove that if $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then f can be written as $f = g + h$, where g is absolutely continuous and h is singular, which are unique up to additive constants. (10%)

4. Prove that if $f \in L^p(E)$ and $f\geq 0$, then $\int_E f^p = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$, where ω is the distribution function of f, defined by $\omega(\alpha) = \left|\{x \in E \mid f(x) > \alpha\}\right|$. (10%)

5. Prove that if $f \in L^p(\mathbb{R})$, where $1 \leq p < \infty$, then for every $\varepsilon > 0$ there is a continuous function g with compact support such that $\|f-g\|_p < \varepsilon$. (10%)

6. Prove that if $f\in L(\mathbb{R}^n)$, then the definite integral $F(E) = \int_E f(x)\,dx$ is absolutely continuous with respect to Lebesgue measure. (10%)

7. For $f, g \in L(\mathbb{R}^n)$, we define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \quad \text{for } x \in \mathbb{R}^n .$$

Prove that $f * g \in L(R^n)$, and $\|f * g\|_1 \le \|f\|_1 \cdot \|g\|_1$. (10%)

8. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0,1]$, and $\{c_k\}$ be a sequence in $\ell^2(R)$. Prove that there exists $f \in L^2[0,1]$ such that $\sum_{k=1}^{\infty} c_k \varphi_k(x)$ is the Fourier series of f with respect to the orthonormal system $\{\varphi_k\}$. (10%)

103 學年度數學系博士班資格考試

(實 變 分 析)

2015. 4. 30

※ 本試題卷共 8 題計算證明題

1. (a) Prove that if every measurable set E in $\mathbb{R}^n$ can be expressed as $E = F \cup Z$, where F is a closed set and $|Z| = 0$.

   (b) Let E_1 and E_2 be measurable subsets of $\mathbb{R}^n$. Prove that the product set $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1| \cdot |E_2|$.

(15%)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Prove that the function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $g(x, y) = f(x - y)$ is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. (10%)

   Hint : Show that there exists an invertible (2×2) matrix A such that
   $$\{ (x, y) \mid g(x, y) > a \} = A\left(\mathbb{R}^n \times \{ z \mid f(z) > a \}\right) \text{ for all } a \in \mathbb{R}.$$

3. Prove or disprove (Please explain your answer):

   (a) There exists a Riemann integrable function $f : [0,1] \to [0,1]$ such that f is continuous at each rational point and discontinuous at each irrational point of $[0,1]$.

   (b) There exists an increasing continuous function f whose derivative f' is Lebesgue integrable on $[0,1]$ such that $\int_{[0,1]} f' \neq f(1) - f(0)$. (10%)

4. (a) Prove carefully that for $0 < a < b < \infty$, $\int_{[0,\infty)} \int_{[a,b]} e^{-xy} \sin x \, dx \, dy = \int_{[a,b]} \frac{\sin x}{x} dx$.

   (b) Evaluate the Lebesgue integral $\int_{(0,\infty)} \frac{\sin x}{x} dx$. (15%)

5. Let $f : [0,1] \to \mathbb{R}$ be measurable. Prove that if $g(x, y) = f(x) - f(y)$ is Lebesgue integrable over $[0,1] \times [0,1]$, then f is Lebesgue integrable on $[0,1]$.

(10%)

6. Let $f_k : E \to \mathbb{R}$ be a sequence of measurable functions on E, where E is a measurable subset of $\mathbb{R}^n$, and $1 \le p < \infty$.

(a) State the definition that $\langle f_k \rangle$ converges to f in measure.

(b) State the definition that $\langle f_k \rangle$ converges to f in L^p.

(c) Prove that if $\langle f_k \rangle$ converges to f in L^p, then it converges to f in measure.

(15%)

7. (a) State without proof Holder inequality.

(b) Let E be a measurable subset of $\mathbb{R}^n$, with $|E| \le 1$, and $1 \le p < q < \infty$. Prove that for any measurable function $f : E \to \mathbb{R}$, $\|f\|_p \le \|f\|_q$.

(10%)

8. (a) Let $f \in L^2(0,1)$. Prove that $\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx \, dx = 0$.

(b) Is (a) still true if $f \in L^1(0,1)$? Why?

(15%)

104學年度數學系博士班資格考試

(實 變 分 析)

2015. 10. 30

※ 本試題卷共六大題(第一大題50分，其餘各題每題10分)

1. Prove or disprove： (Please explain your answer)

(1) There is a Lebesgue measurable subset in $\mathbb{R}^n$, which is not Borel measurable.

(2) Any function f of bounded variation on $[a,b]$ is Riemann integrable .

(3) There is a subset E of $\mathbb{R}$, with $|E|_e > 0$, satisfying for any $x,\ y \in E$ with $x \neq y$, $x - y$ is not a rational number.

(4) There is a sequence $\{E_k\}$ of disjoint sets such that $\left|\bigcup_{k=1}^{\infty} E_k\right|_e < \sum_{k=1}^{\infty} |E_k|_e$.

(5) If $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable, then the function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $g(x,\ y) = f(x - y)$ is also Lebesgue measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

(6) Every Riemann integrable function $f : [0,1] \to \mathbb{R}$ is Lebesgue integrable.

(7) If f is Lebesgue integrable over E, then f is finite a.e. in E .

(8) If $1 \leq p < q < \infty$, then $L^q[0,1] \subset L^p[0,1]$.

(9) There exists an increasing continuous function f whose derivative f' is Lebesgue integrable on $[0,1]$ such that $\int_{[0,1]} f' \neq f(1) - f(0)$.

(10) Any function f of bounded variation on $[a,b]$ can be written as $f = g + h$, where g is absolutely continuous and h is singular.

(50%)

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine function defined by $T(x) = Ax + u$, where A is an $n \times n$ matrix, and u is a fixed vector in $\mathbb{R}^n$. Prove that for any Lebesgue measurable set E of $\mathbb{R}^n$, $|T(E)| = |\det A||E|$. (10%)

3. Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function, where E is a Lebesgue measurable subset of $\mathbb{R}^n$ with $|E| < \infty$. Prove that there exists a sequence $\langle f_k \rangle$ of simple measurable functions on E such that $\langle f_k \rangle$ converges almost uniformly to f in the following sense: for all $\varepsilon > 0$, there exists a closed subset F of E with $|E \setminus F| < \varepsilon$, such that $\langle f_k \rangle$ converges uniformly to f on F. (Hint: You can apply Egorov Theorem) (10%)

4. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ satisfy for each $x \in [0, 1]$, $f(x, y)$ is a Lebesgue integrable function of y, and $\dfrac{\partial f(x, y)}{\partial x}$ is a bounded function of (x, y). Prove that $\dfrac{\partial f(x, y)}{\partial x}$ is a Lebesgue measurable function of y for each $x \in [0, 1]$, and

$$\frac{d}{dx}\int_{[0,1]} f(x, y)\, dy = \int_{[0,1]} \frac{\partial f(x, y)}{\partial x}\, dy. \qquad (10\%)$$

5. Let f be nonnegative and Lebesgue measurable on a Lebesgue measurable subset E of $\mathbb{R}^n$. Prove that

$$\int_E f = \sup \sum_j [\inf_{x \in E_j} f(x)]\,|E_j| \ ,$$

where the supremum is taken over all decompositions $E = \bigcup_j E_j$ of E into the union of a finite number of disjoint Lebesgue measurable sets E_j. (10%)

6. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$, and $\{c_k\}$ be a sequence in $\ell^2(\mathbb{R})$. Prove that there exists $f \in L^2[0, 1]$ such that $\sum_{k=1}^{\infty} c_k \varphi_k(x)$ is the Fourier series of f with respect to the orthonormal system $\{\varphi_k\}$. (10%)

105 學年度數學系博士班資格考試

(Real Analysis Qualifying Exam) **2016.10.31**

1. Let E, F be measurable sets in $\mathbb{R}^n$, B be a Borel set in $[0,\infty)$, and $f: E \to [0,\infty)$ be a measurable function. Prove that the following 4 sets are measurable:
$$E \cup F,\ E \times F,\ f^{-1}\{B\},\ \text{and}\ R(f,E) = \{(x,y) \mid x \in E, 0 \le y \le f(x)\}. \qquad (20\%)$$

2. (a) Use Caratheodory theorem to show that if E is a subset of $\mathbb{R}^n$ satisfying the condition $|G| = |G \cap E|_e + |G \cap E^C|_e$ for all open sets G in $\mathbb{R}^n$, then E is measurable.
(b) If the condition in (a) is changed to $|F| = |F \cap E|_e + |F \cap E^C|_e$ for all closed sets F in $\mathbb{R}^n$, is E measurable? Why? (10%)

3. Prove that if $f: \mathbb{R}^n \to \mathbb{R}$ is a measurable function, then the function $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $g(x, y) = f(2x - 3y)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. (10%)
(Hint: Find an invertible (2×2) matrix A such that
$$\{ (x, y) \mid g(x, y) > a \} = A\left(\mathbb{R}^n \times \{ z \mid f(z) > a \}\right) \text{ for every } a \in \mathbb{R}.)$$

4. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E of $\mathbb{R}^n$.
(a) Use monotone convergence theorem to show that $\int_E \sum_{k=1}^{\infty} |f_k| = \sum_{k=1}^{\infty} \int_E |f_k|$.
(b) Prove that if the series $\sum_{k=1}^{\infty} \int_E |f_k|$ converges, then $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E, and $\sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k$. (16%)

5. (a) Prove that if $f \in L(E)$, then for all $\varepsilon > 0$, there is $\delta > 0$ such that $\int_A |f| < \varepsilon$ for all measurable subsets A of E with $|A| < \delta$.
(b) Use Egoroff theorem to show that if $\langle f_k \rangle$ is a sequence of measurable functions that converges to f a.e. in E, with $|E| < \infty$, and $\sup_k |f_k - f| \in L(E)$, then $\lim_{k\to\infty} \int_E f_k = \int_E f$.
(c) Use Tonelli theorem to show that if $f, g \in L(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} |f(x-y) \times g(y)|\, dy < \infty$ for a.e. $x \in \mathbb{R}^n$. (24%)

6. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$. Prove that $\{\varphi_k\}$ is complete if, and only if, Parseval's formula $\|f\| = \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}$ holds for every $f \in L^2[0, 1]$, where the numbers c_k are the Fourier coefficients of f with respect to the system $\{\varphi_k\}$. (10%)

7. Use Radon-Nikodym theorem to show that for any continuous linear functional T on $L^2[0, 1]$, there exists a unique function $g \in L^2[0, 1]$ such that $T(f) = \int_{[0,1]} f \times g$ for every $f \in L^2[0, 1]$. (10%)

106 學年度數學系博士班資格考試

(Real Analysis Qualifying Exam) **2017.10.31**

*****Each problem is worth 10 points.*****

1. Determine which function is Riemann (improper) integrable on E ? Lebeague integrable on E ? Explain your answer.

$$f(x)=\begin{cases}1, & \text{if } x\in[0,1]\cap\mathbb{Q}\\ x, & \text{if } x\in[0,1]\cap\mathbb{Q}^C\end{cases} \text{ on } E=[0,1] \text{ and } g(x)=\frac{\sin x}{x} \text{ on } E=[1,\infty).$$

2. Prove that (Caratheodory Theorem) a subset E in $\mathbb{R}^n$ is measurable if and only if for every set A in $\mathbb{R}^n$, $|A|_e=|A\cap E|_e+|A\setminus E|_e$.

3. Construct a sequence of disjoint sets $E_1,E_2,E_3,\cdots$ in $\mathbb{R}$ such that $\left|\bigcup_{k=1}^{\infty}E_k\right|_e\neq\sum_{k=1}^{\infty}|E_k|_e$.

4. Prove that there exists a Lebesgue measurable set in $\mathbb{R}$, which is not a Borel set.

5. Prove that if $f:\mathbb{R}^n\to\mathbb{R}$ is measurable, then the function $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, defined by $g(x,y)=f(x+2y)$, is also measurable on $\mathbb{R}^n\times\mathbb{R}^n$.

6. Let $\langle f_k\rangle$ be a sequence of measurable functions on a measurable set E of $\mathbb{R}^n$. Prove that if the series $\sum_{k=1}^{\infty}\int_E|f_k|$ converges, then $\sum_{k=1}^{\infty}f_k$ converges absolutely $a.e.$ in E, and $\sum_{k=1}^{\infty}\int_E f_k=\int_E\sum_{k=1}^{\infty}f_k$.

7. Suppose that $f\in L(\mathbb{R})$ and $\iint_{\mathbb{R}^2}f(3x)f(x+2y)\,dx\,dy=1$, calculate $\int_{\mathbb{R}}f(x)\,dx$.

8. (a) Prove that if $f:[a,b]\to\mathbb{R}$ is bounded, Lebesgue integrable, and $F(x)=\int_{[a,x]}f$, then F is absolutely continuous, and $F'=f$ $a.e.$ in $[a,b]$.

   (b) Is (a) still true, if f is unbounded? Why?

9. Let $f\in L^p(\mathbb{R}^n)$, $1<p,q<\infty$, and $\frac{1}{p}+\frac{1}{q}=1$. Prove that $\|f\|_p=\sup_{\|g\|_q\le1}\left|\int_{\mathbb{R}^n}f(x)\times g(x)\,dx\right|$.

10. (a) Let $f\in L^2(0,2\pi)$. Prove that $\lim_{k\to\infty}\int_0^{2\pi}f(x)\cos kx\,dx=\lim_{k\to\infty}\int_0^{2\pi}f(x)\sin kx\,dx=0$.

    (b) Is (a) still true, if $f\in L^1(0,2\pi)$? Why?

108 學年度數學系博士班資格考試(實變分析)

Real Analysis Qualifying Exam **2019.10.31**

1. It is known from Caratheodory theorem that a subset E of $\mathbb{R}^n$ is measurable if and only if $|A| = |A \cap E|_e + |A \setminus E|_e$ for all sets A in $\mathbb{R}^n$. Prove or disprove :
   (a) If $|G| = |G \cap E|_e + |G \setminus E|_e$ for all open sets G in $\mathbb{R}^n$, then E is measurable.
   (b) If $|F| = |F \cap E|_e + |F \setminus E|_e$ for all closed sets F in $\mathbb{R}^n$, then E is measurable.

(12%)

2. (a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function and B denote the Borel σ-algebra in $\mathbb{R}$. Prove that the family $\Gamma = \{E \subset \mathbb{R} \mid f^{-1}(E) \text{ is measurable}\}$ is a σ-algebra containing B.
   (b) Prove that there exists a measurable subset of $[0,1]$, but not a Borel set.

(12%)

3. (a) Prove that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps measurable subsets of $\mathbb{R}^n$ into measurable sets.
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function, and $a, b \in \mathbb{R}$. Prove that the function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $g(x, y) = f(ax + by)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

(12%)

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then f must be linear. (10%)

5. (a) Prove that if $f \in L(E)$, then f is finite *a.e.* in E.
   (b) Suppose that $\langle f_k \rangle$ is a sequence of measurable functions on a measurable subset E of $\mathbb{R}^n$, and $\sum_{k=1}^{\infty} \int_E |f_k|$ converges. Prove that $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E, and $\sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k$. (12%)

6. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable subset E of $\mathbb{R}^n$, with $|E| < \infty$, and $|f_k(x)| \le M_x < \infty$ for all k and for each $x \in E$. Prove that for all $\varepsilon > 0$, there is a closed subset F of E and a positive number M such that $|E \setminus F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and for all $x \in F$. (**Hint :** You can apply Lusin theorem) (10%)

7. Use Tonelli theorem to show that if $f: E \to [0,\infty)$ is a measurable function on a measurable subset E of $\mathbb{R}^n$, and $\omega(\alpha) = \left|\left\{ x \in E \mid f(x) > \alpha \right\}\right|$, then $\int_E f = \int_0^\infty \omega(\alpha)\,d\alpha$.

(**Hint :** $\int_E f = \iint_{R(f,E)} 1\,dx\,dy$, where $R(f,E) = \{(x,y) \mid x \in E, 0 \le f(x) \le y\}$.) (10%)

8. Let $f:[0,1]\times[0,1] \to \mathbb{R}$ be a measurable function. Prove that if the iterated integral $\int_{[0,1]}\int_{[0,1]} |f(x,y)|\,dx\,dy$ exists and is finite, then $f \in L([0,1]\times[0,1])$, and

$$\iint_{[0,1]\times[0,1]} f = \int_{[0,1]}\int_{[0,1]} f(x,y)\,dx\,dy = \int_{[0,1]}\int_{[0,1]} f(x,y)\,dy\,dx. \quad (10\%)$$

9. Let $\{\varphi_k\}$ be any orthonormal basis for $L^2(E)$ over $\mathbb{R}$.

(a) Prove that $\{\varphi_k\}$ must be countable and complete.

(b) Prove that any function $f \in L^2(E)$ satisfies Parseval formula with respect to $\{\varphi_k\}$; that is, $\|f\|_2 = \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{\frac{1}{2}}$, where $\{c_k\}$ is the sequence of Fourier coefficients of f.

(12%)

109 學年度數學系博士班資格考試(實變分析)

Real Analysis Qualifying Exam

2021.4.28

1. Let $f(x)=\begin{cases}0, & \text{if } x\in[0,1]\\ 1, & \text{if } x\in(1,2]\end{cases}$, $\alpha(x)=\begin{cases}0, & \text{if } x\in[0,1)\\ 1, & \text{if } x\in[1,2]\end{cases}$, and $\beta(x)=\begin{cases}x, & \text{if } x\in[0,1)\\ x^2, & \text{if } x\in[1,2]\end{cases}$.

   (a) Is f Riemann-Stieltjes integrable to α on $[0,2]$? Why?

   (b) Is f Riemann-Stieltjes integrable to β on $[0,2]$? Why? (12%)

2. (a) Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be a measurable function and B be a Borel set in $\mathbb{R}$. Prove that $f^{-1}(B)$ is measurable in $[0,1]\times[0,1]$.

   (b) Let f and g be measurable on $[0,1]$. Prove that the function $F:[0,1]\times[0,1]\to\mathbb{R}$, defined by $F(x,y)=f(x)\times g(y)$, is measurable on $[0,1]\times[0,1]$. (12%)

3. Let $f:E\to\mathbb{R}$ be a measurable function on a measurable subset E of $\mathbb{R}^n$. Prove that for all $\varepsilon>0$, there is a Borel set B in E, with $|E\setminus B|<\varepsilon$, and a sequence $\langle f_k\rangle$ of Borel measurable functions such that $\langle f_k(x)\rangle$ converges increasingly to $|f(x)|$ for all $x\in B$.

   (10%)

4. Let $\langle f_k\rangle$ be a sequence of measurable functions on a measurable subset E of $\mathbb{R}^n$, and $\sum_{k=1}^{\infty}\int_E |f_k|$ converges. Prove that $\sum_{k=1}^{\infty}|f_k|$ converges *a.e.* in E, and $\sum_{k=1}^{\infty}\int_E f_k = \int_E \sum_{k=1}^{\infty} f_k$.

   (10%)

5. Let $\langle f_k\rangle$ be a sequence of increasing functions on $[a,b]$, and $\sum_{k=1}^{\infty} f_k(x)$ converge to $f(x)$ for each $x\in[a,b]$. Prove that $\sum_{k=1}^{\infty} f_k'(x)$ converges to $f'(x)$ for *a.e.* x in E.

   (10%)

6. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfy that for each $x \in [0,1]$, $f(x, y)$ is a Lebesgue integrable function of y, and $\dfrac{\partial f(x, y)}{\partial x}$ is a bounded function of (x, y). Prove that $\dfrac{\partial f(x, y)}{\partial x}$ is a measurable function of y for each $x \in [0,1]$, and

$$\frac{d}{dx}\int_{[0,1]} f(x, y)\, dy = \int_{[0,1]} \frac{\partial f(x, y)}{\partial x}\, dy.$$

(10%)

7. Let E be a measurable subset of $\mathbb{R}^n$. Prove that $f : E \to \mathbb{R}$ is measurable if and only If the region $R(f, E)$ is measurable, where $R(f, E) = \{(x, y) \mid x \in E, 0 \le f(x) \le y\}$.

(12%)

8. (a) Let f be measurable on E, and $1 < p < q < \infty$, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Prove that

$$\int_E |fg| \le \left(\int_E |f|^p\right)^{\frac{1}{p}} \left(\int_E |f|^q\right)^{\frac{1}{q}}$$

(b) Let f be measurable on E with $0 < |E| < \infty$, and $1 \le p < q < \infty$. Prove that

$$\left(\frac{1}{|E|}\int_E |f|^p\right)^{\frac{1}{p}} \le \left(\frac{1}{|E|}\int_E |f|^q\right)^{\frac{1}{q}}.$$

(12%)

9. Define the operator $T : C[0,1] \to \mathbb{R}$ by $T(f) = f(1)$ for all $f \in C[0,1]$, where $C[0,1]$ denotes the Banach space of all real-valued continuous functions on $[0,1]$.

(a) Prove that T is a continuous linear functional on $C[0,1]$.

(b) Prove that there exists a continuous linear functional $T^* : L^\infty[0,1] \to \mathbb{R}$ such that $T^*(f) = T(f)$ for all $f \in C[0,1]$, but there exists no function $g \in L^1[0,1]$ satisfying $T^*(f) = \int_{[0,1]} (f \times g)\, dx$ for all $f \in C[0,1]$. (12%)

Real Analysis Qualifying Exam October 27, 2023

Fall 112.

English Name: ____________________________

Chinese Name: ____________________________

Grading. The exam is out of 100pts. As written below, Problems 1, 2, 6, 7, 8 are worth 12 pts; Problems 4, 5 are worth 13 pts; Problem 3 is worth 14 pts.

Preliminaries. Throughout this exam, we suppose X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, elements of which we call measurable, and $\mu : \mathcal{B} \to [0, \infty]$ is a measure:

i
$$\mu(\emptyset) = 0;$$

ii
$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n) \quad E_n \in \mathcal{B} \text{ for all } n, E_n \cap E_m = \emptyset \text{ if } m \neq n.$$

Further suppose $X = \cup_n X_n$ with $\mu(X_n) < +\infty$. We say that a function $f : X \to [-\infty, \infty]$ is measurable if $\{x : f(x) > \alpha\} \in \mathcal{B}$ for each $\alpha \in \mathbb{R}$. For a measurable function $f : X \to [0, \infty]$ define

$$\int_X f \, d\mu := \sup_{g \leq f} \int_X g \, d\mu$$

where the supremum is taken over all non-negative simple functions.

1 (12 pts). Suppose that $f : X \to \mathbb{R}$ is a measurable function such that

$$\int_X |f| \, d\mu < +\infty.$$

Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if A is a measurable set with $\mu(A) < \delta$ then

$$\int_A |f| \, d\mu < \epsilon. \tag{1}$$

2 (12 pts). Show that if $\{A_n\}$ is a sequence of measurable sets with $A_{n+1} \subset A_n$ and $\mu(A_1) < +\infty$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=0}^{\infty} A_n). \tag{2}$$

3 (14 pts). Show that if

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$, then

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu. \tag{3}$$

(If you utilize Egorov's theorem, monotone convergence theorem, dominated convergence theorem, etc. in your proof you should prove them first.)

4 (13 pts). In this problem, let $X = \mathbb{R}^n$ and suppose μ is a Radon measure, i.e. finite on compact sets and for each measurable set satisfies

$$\mu(E) = \sup_{K \subset E} \mu(K) = \inf_{U \supset E} \mu(U)$$

where K are assumed to be compact and U open. Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is integrable then there exists a sequence of continuous functions φ_n such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |\varphi_n - f| \, dx = 0. \tag{4}$$

5 (13 pts). In this problem, let $X = \mathbb{R}^n$ and suppose μ is a Radon measure, i.e. finite on compact sets and for each measurable set satisfies

$$\mu(E) = \sup_{K \subset E} \mu(K) = \inf_{U \supset E} \mu(U)$$

where K are assumed to be compact and U open. Define the Hardy-Littlewood maximal function of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ which is integrable on compact subsets by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, d\mu.$$

Suppose for the given μ that one has shown the weak-type estimate

$$\mu(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f| \, d\mu.$$

Use this estimate and the properties of μ to show that

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) \, d\mu = f(x) \tag{5}$$

for μ almost every $x \in \mathbb{R}^n$. (You may assume that the conclusion of Problem 4 is valid.)

6 (12 pts). In this problem, let $X = [0,1]$, $\mathcal{B} = \mathcal{M}$ be the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and μ be the Lebesgue measure. Suppose that $f_n, f \in L^2([0,1])$,

$$\lim_{n\to\infty} \int_{[0,1]} f_n g \, dx = \int_{[0,1]} f g \, dx$$

for every $g \in L^2([0,1])$ and that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n|^2 \, dx = \int_{[0,1]} |f|^2 \, dx.$$

Show that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n - f|^2 \, dx = 0. \tag{6}$$

7 (12 pts). In this problem, let $X = [0,1]$, $\mathcal{B} = \mathcal{M}$ be the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and μ be the Lebesgue measure. Suppose that $f_n, f \in L^2([0,1])$ and

$$\lim_{n\to\infty} \int_{[0,1]} |f_n - f|^2 \, dx = 0.$$

Show that there exists a subsequence $\{f_{n_k}\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \tag{7}$$

for Lebesgue almost every $x \in [0,1]$.

8 (12 pts). Let ν be another measure on the measurable space $(X, \mathcal{B})$ for which $X = \cup_n X_n'$ with $\nu(X_n') < +\infty$. State the Radon-Nikodym theorem and the Lebesgue decomposition theorem for the measures μ, ν, introducing suitable hypothesis when necessary.