Total score (100 points) ：

1 *(10 points) Find the general solution of* $x' = Ax$ *with*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}.$$

Here, eigenvalues of the matrix A *are* 1, 1, *and* 0.

2 *(a) (5 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\frac{dx}{dt} = y$$
$$\frac{dt}{dt} = -x - \mu(x^2 - 1)y$$

(b) (15 points) Find the stabilities of the critical point $(0, 0)$ *for the cases* $\mu > 0$ *and* $\mu < 0$.

3 *(15 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. *Let* $y = x'$. *Show that* $(0, 0)$ *is Liapunov stable using Liapunov function of the form* $V(x, y) = \alpha x^2 + \beta x^4 + \gamma y^2$ *for* $\alpha, \beta, \gamma > 0$.

4 *(15 points) Consider the following system*

$$\frac{dx}{dt} = x - y - x^3$$
$$\frac{dt}{dt} = x + y - y^3$$

Show that there is at least one stable limit cycle in the region $A = \{(x, y) \in \mathbb{R}^2 | 1 \leq |(x, y)| \leq \sqrt{2}\}$.

5 *(15 points) If* $C \geq 0$ *and* $u, v : [0, \beta] \to [0, \infty)$ *are continuous and*

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

for all $t \in [0, \beta]$*, then*

$$u(t) \leq Ce^{v(t)},$$

where $v(t) = \int_0^t v(s)ds$.

6 *(a) (15 points) Let* $n = 2$*. For any* 2×2 *constant real matrix* A*, show that there exists an invertible real matrix* P *such that the matrix*

$$B = P^{-1}AP$$

has one of the following forms

$$\text{(i)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ (ii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ (iii)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $\lambda, \mu, a, b \in \mathbb{R}$*. Find* P *explicitly.*

(b) (10 points) Let $A = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$*. where* $\lambda, \mu, \alpha \in \mathbb{R}$*. Solve the initial value problem:* $x' = Ax,\ x(0) = x_0$.