# Geometry and Topology Qualifying Examination
# Fall 2020

October 30, 2020

1. Let $(M^n, g)$ be a Riemannian manifold.
   (a) Define the Levi-Civita connection on $M$. (5%)
   (b) Let $p \in M$ and $U$ be a local chart of $p$. Suppose $(x_1, \cdots, x_0)$ is the local coordinate on $U$. Define the Christoffel symbols $\Gamma_{ij}^k (i, j, k = 1, \cdots, n)$ by
   $$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$
   Show that $\Gamma_{ij}^k = \Gamma_{ji}^k$. (10%)
   (c) Define the Riemann curvature $R(X, Y)Z$ for any $X, Y, Z \in TM$. (5%)
   (d) State and prove the Bianchi Identity of the Riemann curvature tensor $R$. (10%)

2. Let $M$ be a Riemannian manifold and $f \in C^3(M)$. Suppose $\{x_i\}_{i=1}^n$ is a normal coordinate system at $p \in M$. Derive the following Bochner formula:
   $$\tfrac{1}{2}\Delta|\nabla f|^2 = \sum_{i,j} |f_{ij}|^2 + \sum_{i,j} R_{ij} f_i f_j + \sum_i f_i (\Delta f)_i$$
   where $f_i$ denotes the derivative of $f$ with respect to $\frac{\partial}{\partial x_i}$ and $R_{ij}$ is the Ricci curvature. (15%)

3. Let $M^2 \subseteq \mathbb{R}^3$ be an embedded compact, closed surface of genus $\geq 1$. Show that the Gaussian curvature of $M$ must be vanish somewhere on $M$. (15%)

4. Consider the torus of revolution $T$ obtained by rotating the circle $(x-a)^2 + Z^2 = r^2$ around z-axis:
   $$T = \{(x, y, z) | (x^2 + y^2 + z^2 + a^2 - r^2)^2 - 4a^2(x^2 + y^2) = 0\}$$
   Parametrize this torus, compute its Gaussian curvature function $K$, and verify $\int_T K dA = 0$ by explicit calculation. (20%)

5. Let $(M, g)$ be a Riemannian manifold. Let $p \in M$ and the map $exp_p : B_\varepsilon(0) \subset T_pM \to M$ is the exponential map at $p$ which is always defined in a small neighborhood $B_\varepsilon(0)$ of the origin of $T_pM$. Show that there exists a $\delta > 0$ such that
   $$exp_p : B_\delta(0) \subset T_pM \to M$$
   is a diffeomorphism onto its image. (10%)

6. Show that there does not exist any nonconstant harmonic function on a compact, connected Riemannian manifold without boundary. (10%)