

100 學年度下學期數學系博士班資格考試
(實變分析)

本試題卷共 2 頁，計 10 題計算證明題，每題 10 分，合計 100 分。

1. Prove the *Carathéodory theorem*: A set E is measurable if and only if for every set A ,

$$|A|_e = |A \cap E|_e + |A \setminus E|_e.$$

(Note: $|A|_e$ denotes the outer measure of A .)

2. Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.
3. Let f be a function which is upper semi-continuous and finite on a compact set E . Show that if f is bounded above on E . Show also that f assumes its maximum on E , that is, that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.
4. Let $f \in L(0, 1)$. Show that $x^k f(x) \in L(0, 1)$ for $k = 1, 2, \dots$, and $\int_0^1 x^k f(x) dx \rightarrow 0$ as $k \rightarrow \infty$.
5. Let E be a measurable subset of \mathbb{R}^2 such that for almost every $x \in \mathbb{R}^1$, $\{y \mid (x, y) \in E\}$ has \mathbb{R}^1 -measure zero. Show that E has measure zero, and the for almost every $y \in \mathbb{R}^1$, $\{x \mid (x, y) \in E\}$ has measure zero.
6. (a) Write out the definition of the essential supremum $\|f\|_\infty$ of a real-valued measurable function f on a measurable set E .
(b) Let f be a real-valued measurable function on $[0, 1]$. Prove that $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.
7. Let E be a measurable set in \mathbb{R}^n , and $0 < p < q \leq \infty$.
(a) Prove that $L^p(E) \cap L^\infty(E) \subset L^q(E)$.
(b) Prove that if $|E| < \infty$, then $L^q(E) \subset L^p(E)$.
8. Let $f, g \in L^2(\mathbb{R}^n)$. Prove that $f + g \in L^2(\mathbb{R}^n)$ and $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$.

(背面尚有試題)

9. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$, and $\{c_k\}$ be the Fourier series of a function $f \in L^2[0, 1]$ with respect to the system $\{\varphi_k\}$.

(a) Prove that the Bessel's inequality $\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} \leq \|f\|_2$ holds.

(b) Find a necessary and sufficient condition so that the Parseval's identity $\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} = \|f\|_2$ holds, and prove your answer.

10. Let $C[0, 1]$ denote the set of all real-valued continuous functions on $[0, 1]$, and the linear operator $T : C[0, 1] \rightarrow \mathbb{R}$ be defined by $T(f) = f(1)$ for all $f \in C[0, 1]$.

(a) Prove that T is a continuous linear functional on $C[0, 1]$.

(b) Prove that there exists an extension $T^* : L^\infty[0, 1] \rightarrow \mathbb{R}^n$ of T such that T^* is a continuous linear functional on $L^\infty[0, 1]$, but there is no $g \in L^1[0, 1]$ satisfying

$$T^*(f) = \int_{[0,1]} (f \times g) dx \quad \text{for all } f \in C[0, 1].$$

(試題結束)

101 學年度上學期數學系博士班資格考試
(實變分析)

本試題卷共 2 頁，計 10 題計算證明題，每題 10 分，合計 100 分。

1. Let E be a measurable subset of \mathbb{R} , with $|E| > 0$. Prove that there exists a positive real number ε such that $(-\varepsilon, \varepsilon) \subset E - E$, where

$$E - E = \{x - y \mid x, y \in E\}.$$

2. Prove or disprove:

- (a) Any function $f : [a, b] \rightarrow \mathbb{R}$ of bounded variation is measurable.
(b) Any upper semicontinuous function $f : [a, b] \rightarrow \mathbb{R}$ is measurable.

3. Let E be a measurable set in \mathbb{R}^n of finite measure. Prove that $f : E \rightarrow \mathbb{R}$ is measurable if and only if for any $\varepsilon > 0$, there exists a closed subset F of E such that $|E \setminus F| < \varepsilon$, and f is continuous on F .

4. (a) State without proof the Egorov's theorem.

- (b) Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E with $|E| < \infty$. If f_k converges to f a.e. in E , and $\sup_k |f_k - f| \in L(E)$, prove that $\lim_{k \rightarrow \infty} \int_E f_k = \int_E f$.

5. Let $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ satisfy for each $x \in [0, 1]$, $f(x, y)$ is a Lebesgue integrable function of y , and $\frac{\partial f(x, y)}{\partial x}$ is a bounded function of (x, y) . Prove that $\frac{\partial f(x, y)}{\partial x}$ is a measurable function of y for each $x \in [0, 1]$, and

$$\frac{d}{dx} \int_{[0,1]} f(x, y) dy = \int_{[0,1]} \frac{\partial f(x, y)}{\partial x} dy.$$

6. (a) State the definition for a finite function f on a finite interval $[a, b]$ to be *absolutely continuous*.

- (b) Show that the function $f(x) = x^\alpha$ is absolutely continuous on every bounded subinterval of $[0, \infty)$ whenever $\alpha > 0$.

7. Let a_1, a_2, \dots, a_N be non-negative real numbers, p_1, p_2, \dots, p_N be positive real numbers with $\sum_{j=1}^N (1/p_j) = 1$. Show that

$$\prod_{j=1}^N a_j \leq \sum_{j=1}^N \frac{a_j}{p_j}.$$

(背面尚有試題)

8. Let ℓ^∞ denote the normed linear space of all bounded real sequences. Is ℓ^∞ separable? Justify your answer.
9. Suppose that $f_k, f \in L^2$, and that $\int f_k g \rightarrow \int f g$ for all $g \in L^2$. If $\|f_k\|_2 \rightarrow \|f\|_2$, show that $f_k \rightarrow f$ in L^2 norm.
10. Let Σ be a σ -algebra on a set \mathcal{S} , $\{E_k\}$ be any sequence of sets in Σ , and ϕ be a non-negative additive set function on Σ . Prove that

$$\phi\left(\liminf_{k \rightarrow \infty} E_k\right) \leq \liminf_{k \rightarrow \infty} \phi(E_k).$$

(試題結束)

103 學年度數學系博士班資格考試
(實變分析)

※ 本試題卷共 8 題證明題

1. (a) Prove that every Borel measurable subset in \mathbb{R}^n is Lebesgue measurable.
(b) Prove that there is a Lebesgue measurable subset in \mathbb{R}^n is not Borel measurable.
(10%)

2. Prove or disprove (Please explain your answer):
 - (a) If $f : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, then f is Lebesgue measurable.
 - (b) If E is a Lebesgue measurable subset of \mathbb{R} , with $|E| > 0$, then there exist $x, y \in E$ with $x \neq y$ such that $x - y$ is a rational number.
 - (c) If for each rational number a , the set $\{x \in \mathbb{R}^n \mid f(x) > a\}$ is Lebesgue measurable, then $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lebesgue measurable.
 - (d) There exists a Riemann integrable function $f : [0, 1] \rightarrow [0, 1]$ such that f is continuous at each rational point and discontinuous at each irrational point of $[0, 1]$.
 - (e) If f is Lebesgue integrable over E , then f is finite a.e. in E .
(30%)

3. Prove that if $f : [a, b] \rightarrow \mathbb{R}$ is a function of bounded variation, then f can be written as $f = g + h$, where g is absolutely continuous and h is singular, which are unique up to additive constants.
(10%)

4. Prove that if $f \in L^p(E)$ and $f \geq 0$, then $\int_E f^p = p \int_0^\infty \alpha^{p-1} \omega(\alpha) d\alpha$, where ω is the distribution function of f , defined by $\omega(\alpha) = |\{x \in E \mid f(x) > \alpha\}|$.
(10%)

5. Prove that if $f \in L^p(\mathbb{R})$, where $1 \leq p < \infty$, then for every $\varepsilon > 0$ there is a continuous function g with compact support such that $\|f - g\|_p < \varepsilon$.
(10%)

6. Prove that if $f \in L(\mathbb{R}^n)$, then the definite integral $F(E) = \int_E f(x) dx$ is absolutely continuous with respect to Lebesgue measure.
(10%)

7. For $f, g \in L(\mathbb{R}^n)$, we define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy \text{ for } x \in \mathbb{R}^n .$$

Prove that $f * g \in L(\mathbb{R}^n)$, and $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$. (10%)

8. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$, and $\{c_k\}$ be a sequence in $\ell^2(\mathbb{R})$. Prove that

there exists $f \in L^2[0, 1]$ such that $\sum_{k=1}^{\infty} c_k \varphi_k(x)$ is the Fourier series of f with respect to the orthonormal system $\{\varphi_k\}$. (10%)

103 學年度數學系博士班資格考試

(實變分析)

2015. 4. 30

※ 本試題卷共 8 題計算證明題

1. (a) Prove that if every measurable set E in \mathbb{R}^n can be expressed as $E = F \cup Z$, where F is a closed set and $|Z| = 0$.

(b) Let E_1 and E_2 be measurable subsets of \mathbb{R}^n . Prove that the product set $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1| \cdot |E_2|$.

(15%)

2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be measurable. Prove that the function $g: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $g(x, y) = f(x - y)$ is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

(10%)

Hint : Show that there exists an invertible (2×2) matrix A such that

$$\{(x, y) \mid g(x, y) > a\} = A(\mathbb{R}^n \times \{z \mid f(z) > a\}) \text{ for all } a \in \mathbb{R}.$$

3. Prove or disprove (Please explain your answer):

(a) There exists a Riemann integrable function $f: [0, 1] \rightarrow [0, 1]$ such that f is continuous at each rational point and discontinuous at each irrational point of $[0, 1]$.

(b) There exists an increasing continuous function f whose derivative f' is Lebesgue integrable on $[0, 1]$ such that $\int_{[0, 1]} f' \neq f(1) - f(0)$.

(10%)

4. (a) Prove carefully that for $0 < a < b < \infty$, $\int_{[0, \infty)} \int_{[a, b]} e^{-xy} \sin x \, dx \, dy = \int_{[a, b]} \frac{\sin x}{x} \, dx$.

(b) Evaluate the Lebesgue integral $\int_{(0, \infty)} \frac{\sin x}{x} \, dx$.

(15%)

5. Let $f: [0, 1] \rightarrow \mathbb{R}$ be measurable. Prove that if $g(x, y) = f(x) - f(y)$ is Lebesgue integrable over $[0, 1] \times [0, 1]$, then f is Lebesgue integrable on $[0, 1]$.

(10%)

6. Let $f_k : E \rightarrow \mathbb{R}$ be a sequence of measurable functions on E , where E is a measurable subset of \mathbb{R}^n , and $1 \leq p < \infty$.

(a) State the definition that $\langle f_k \rangle$ converges to f in measure.

(b) State the definition that $\langle f_k \rangle$ converges to f in L^p .

(c) Prove that if $\langle f_k \rangle$ converges to f in L^p , then it converges to f in measure.

(15%)

7. (a) State without proof Holder inequality.

(b) Let E be a measurable subset of \mathbb{R}^n , with $|E| \leq 1$, and $1 \leq p < q < \infty$. Prove that for any measurable function $f : E \rightarrow \mathbb{R}$, $\|f\|_p \leq \|f\|_q$.

(10%)

8. (a) Let $f \in L^2(0, 1)$. Prove that $\lim_{k \rightarrow \infty} \int_0^{2\pi} f(x) \cos kx \, dx = \lim_{k \rightarrow \infty} \int_0^{2\pi} f(x) \sin kx \, dx = 0$.

(b) Is (a) still true if $f \in L^1(0, 1)$? Why?

(15%)

104 學年度數學系博士班資格考試

(實變分析)

2015. 10. 30

※ 本試題卷共六大題 (第一大題 50 分, 其餘各題每題 10 分)

1. Prove or disprove : (Please explain your answer)

(1) There is a Lebesgue measurable subset in \mathbb{R}^n , which is not Borel measurable.

(2) Any function f of bounded variation on $[a, b]$ is Riemann integrable .

(3) There is a subset E of \mathbb{R} , with $|E|_e > 0$, satisfying for any $x, y \in E$ with $x \neq y$, $x - y$ is not a rational number.

(4) There is a sequence $\{E_k\}$ of disjoint sets such that $\left| \bigcup_{k=1}^{\infty} E_k \right|_e < \sum_{k=1}^{\infty} |E_k|_e$.

(5) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Lebesgue measurable, then the function $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $g(x, y) = f(x - y)$ is also Lebesgue measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

(6) Every Riemann integrable function $f : [0, 1] \rightarrow \mathbb{R}$ is Lebesgue integrable.

(7) If f is Lebesgue integrable over E , then f is finite a.e. in E .

(8) If $1 \leq p < q < \infty$, then $L^q[0, 1] \subset L^p[0, 1]$.

(9) There exists an increasing continuous function f whose derivative f' is Lebesgue integrable on $[0, 1]$ such that $\int_{[0, 1]} f' \neq f(1) - f(0)$.

(10) Any function f of bounded variation on $[a, b]$ can be written as $f = g + h$, where g is absolutely continuous and h is singular.

(50%)

2. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an affine function defined by $T(x) = Ax + u$, where A is an $n \times n$ matrix, and u is a fixed vector in \mathbb{R}^n . Prove that for any Lebesgue measurable set E of \mathbb{R}^n , $|T(E)| = |\det A| |E|$.

(10%)

3. Let $f : E \rightarrow \mathbb{R}$ be a Lebesgue measurable function, where E is a Lebesgue measurable

subset of \mathbb{R}^n with $|E| < \infty$. Prove that there exists a sequence $\langle f_k \rangle$ of simple measurable functions on E such that $\langle f_k \rangle$ converges almost uniformly to f in the following sense: for all $\varepsilon > 0$, there exists a closed subset F of E with $|E \setminus F| < \varepsilon$, such that $\langle f_k \rangle$ converges uniformly to f on F . (Hint: You can apply Egorov Theorem) (10%)

4. Let $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ satisfy for each $x \in [0, 1]$, $f(x, y)$ is a Lebesgue integrable function of y , and $\frac{\partial f(x, y)}{\partial x}$ is a bounded function of (x, y) . Prove that $\frac{\partial f(x, y)}{\partial x}$ is a Lebesgue measurable function of y for each $x \in [0, 1]$, and

$$\frac{d}{dx} \int_{[0,1]} f(x, y) dy = \int_{[0,1]} \frac{\partial f(x, y)}{\partial x} dy. \quad (10\%)$$

5. Let f be nonnegative and Lebesgue measurable on a Lebesgue measurable subset E of \mathbb{R}^n . Prove that

$$\int_E f = \sup \sum_j [\inf_{x \in E_j} f(x)] |E_j|,$$

where the supremum is taken over all decompositions $E = \cup_j E_j$ of E into the union of a finite number of disjoint Lebesgue measurable sets E_j . (10%)

6. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$, and $\{c_k\}$ be a sequence in $\ell^2(\mathbb{R})$. Prove that there exists $f \in L^2[0, 1]$ such that $\sum_{k=1}^{\infty} c_k \varphi_k(x)$ is the Fourier series of f with respect to the orthonormal system $\{\varphi_k\}$. (10%)

105 學年度數學系博士班資格考試

(Real Analysis Qualifying Exam)

2016.10.31

1. Let E, F be measurable sets in \mathbb{R}^n , B be a Borel set in $[0, \infty)$, and $f : E \rightarrow [0, \infty)$ be a measurable function. Prove that the following 4 sets are measurable:

$$E \cup F, E \times F, f^{-1}\{B\}, \text{ and } R(f, E) = \{(x, y) \mid x \in E, 0 \leq y \leq f(x)\}. \quad (20\%)$$

2. (a) Use Caratheodory theorem to show that if E is a subset of \mathbb{R}^n satisfying the condition $|G| = |G \cap E|_e + |G \cap E^C|_e$ for all open sets G in \mathbb{R}^n , then E is measurable.

- (b) If the condition in (a) is changed to $|F| = |F \cap E|_e + |F \cap E^C|_e$ for all closed sets F in \mathbb{R}^n , is E measurable? Why? (10%)

3. Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a measurable function, then the function $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $g(x, y) = f(2x - 3y)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. (10%)

(Hint: Find an invertible (2×2) matrix A such that

$$\{(x, y) \mid g(x, y) > a\} = A \left(\mathbb{R}^n \times \{z \mid f(z) > a\} \right) \text{ for every } a \in \mathbb{R} .)$$

4. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E of \mathbb{R}^n .

(a) Use monotone convergence theorem to show that $\int_E \sum_{k=1}^{\infty} |f_k| = \sum_{k=1}^{\infty} \int_E |f_k|$.

- (b) Prove that if the series $\sum_{k=1}^{\infty} \int_E |f_k|$ converges, then $\sum_{k=1}^{\infty} f_k$ converges absolutely a.e. in

$$E, \text{ and } \sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k. \quad (16\%)$$

5. (a) Prove that if $f \in L(E)$, then for all $\varepsilon > 0$, there is $\delta > 0$ such that $\int_A |f| < \varepsilon$ for all measurable subsets A of E with $|A| < \delta$.

- (b) Use Egoroff theorem to show that if $\langle f_k \rangle$ is a sequence of measurable functions that converges to f a.e. in E , with $|E| < \infty$, and $\sup_k |f_k - f| \in L(E)$, then $\lim_{k \rightarrow \infty} \int_E f_k = \int_E f$.

- (c) Use Tonelli theorem to show that if $f, g \in L(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} |f(x-y) \times g(y)| dy < \infty$ for a.e. $x \in \mathbb{R}^n$. (24%)

6. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$. Prove that $\{\varphi_k\}$ is complete if, and only if,

Parseval's formula $\|f\| = \left(\sum_{k=1}^{\infty} |c_k|^2 \right)^{1/2}$ holds for every $f \in L^2[0, 1]$, where the numbers c_k are the Fourier coefficients of f with respect to the system $\{\varphi_k\}$. (10%)

7. Use Radon-Nikodym theorem to show that for any continuous linear functional T on

$L^2[0, 1]$, there exists a unique function $g \in L^2[0, 1]$ such that $T(f) = \int_{[0,1]} f \times g$ for every $f \in L^2[0, 1]$. (10%)

106 學年度數學系博士班資格考試

(Real Analysis Qualifying Exam)

2017.10.31

Each problem is worth 10 points.

1. Determine which function is Riemann (improper) integrable on E ? Lebesgue integrable on E ? Explain your answer.

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \cap \mathbb{Q} \\ x, & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases} \text{ on } E = [0,1] \text{ and } g(x) = \frac{\sin x}{x} \text{ on } E = [1, \infty).$$

2. Prove that (Caratheodory Theorem) a subset E in \mathbb{R}^n is measurable if and only if for every set A in \mathbb{R}^n , $|A|_e = |A \cap E|_e + |A \setminus E|_e$.

3. Construct a sequence of disjoint sets E_1, E_2, E_3, \dots in \mathbb{R} such that $\left| \bigcup_{k=1}^{\infty} E_k \right|_e \neq \sum_{k=1}^{\infty} |E_k|_e$.

4. Prove that there exists a Lebesgue measurable set in \mathbb{R} , which is not a Borel set.

5. Prove that if $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable, then the function $g: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $g(x, y) = f(x + 2y)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

6. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E of \mathbb{R}^n . Prove that if the series $\sum_{k=1}^{\infty} \int_E |f_k|$ converges, then $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E , and

$$\sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k.$$

7. Suppose that $f \in L(\mathbb{R})$ and $\iint_{\mathbb{R}^2} f(3x)f(x+2y) dx dy = 1$, calculate $\int_{\mathbb{R}} f(x) dx$.

8. (a) Prove that if $f: [a, b] \rightarrow \mathbb{R}$ is bounded, Lebesgue integrable, and $F(x) = \int_{[a,x]} f$,

then F is absolutely continuous, and $F' = f$ *a.e.* in $[a, b]$.

(b) Is (a) still true, if f is unbounded? Why?

9. Let $f \in L^p(\mathbb{R}^n)$, $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\|f\|_p = \sup_{\|g\|_q \leq 1} \left| \int_{\mathbb{R}^n} f(x) \times g(x) dx \right|$.

10. (a) Let $f \in L^2(0, 2\pi)$. Prove that $\lim_{k \rightarrow \infty} \int_0^{2\pi} f(x) \cos kx dx = \lim_{k \rightarrow \infty} \int_0^{2\pi} f(x) \sin kx dx = 0$.

(b) Is (a) still true, if $f \in L^1(0, 2\pi)$? Why?

108 學年度數學系博士班資格考試(實變分析)

Real Analysis Qualifying Exam

2019.10.31

1. It is known from Caratheodory theorem that a subset E of \mathbb{R}^n is measurable if and only if $|A| = |A \cap E|_e + |A \setminus E|_e$ for all sets A in \mathbb{R}^n . Prove or disprove :
- (a) If $|G| = |G \cap E|_e + |G \setminus E|_e$ for all open sets G in \mathbb{R}^n , then E is measurable.
- (b) If $|F| = |F \cap E|_e + |F \setminus E|_e$ for all closed sets F in \mathbb{R}^n , then E is measurable.
- (12%)
2. (a) Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function and B denote the Borel σ -algebra in \mathbb{R} . Prove that the family $\Gamma = \{E \subset \mathbb{R} \mid f^{-1}(E) \text{ is measurable}\}$ is a σ -algebra containing B .
- (b) Prove that there exists a measurable subset of $[0, 1]$, but not a Borel set.
- (12%)
3. (a) Prove that every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ maps measurable subsets of \mathbb{R}^n into measurable sets.
- (b) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function, and $a, b \in \mathbb{R}$. Prove that the function $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $g(x, y) = f(ax + by)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$.
- (12%)
4. Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then f must be linear.
- (10%)
5. (a) Prove that if $f \in L(E)$, then f is finite *a.e.* in E .
- (b) Suppose that $\langle f_k \rangle$ is a sequence of measurable functions on a measurable subset E of \mathbb{R}^n , and $\sum_{k=1}^{\infty} \int_E |f_k|$ converges. Prove that $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E , and $\sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k$.
- (12%)
6. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable subset E of \mathbb{R}^n , with $|E| < \infty$, and $|f_k(x)| \leq M_x < \infty$ for all k and for each $x \in E$. Prove that for all $\varepsilon > 0$, there is a closed subset F of E and a positive number M such that $|E \setminus F| < \varepsilon$ and $|f_k(x)| \leq M$ for all k and for all $x \in F$. (Hint : You can apply Lusin theorem)
- (10%)

7. Use Tonelli theorem to show that if $f : E \rightarrow [0, \infty)$ is a measurable function on a measurable subset E of \mathbb{R}^n , and $\omega(\alpha) = |\{x \in E \mid f(x) > \alpha\}|$, then $\int_E f = \int_0^\infty \omega(\alpha) d\alpha$.

(Hint : $\int_E f = \iint_{R(f,E)} 1 dx dy$, where $R(f,E) = \{(x,y) \mid x \in E, 0 \leq f(x) \leq y\}$.) (10%)

8. Let $f : [0,1] \times [0,1] \rightarrow \mathbb{R}$ be a measurable function. Prove that if the iterated integral

$\int_{[0,1]} \int_{[0,1]} |f(x,y)| dx dy$ exists and is finite, then $f \in L([0,1] \times [0,1])$, and

$$\iint_{[0,1] \times [0,1]} f = \int_{[0,1]} \int_{[0,1]} f(x,y) dx dy = \int_{[0,1]} \int_{[0,1]} f(x,y) dy dx. \quad (10\%)$$

9. Let $\{\varphi_k\}$ be any orthonormal basis for $L^2(E)$ over \mathbb{R} .

(a) Prove that $\{\varphi_k\}$ must be countable and complete.

(b) Prove that any function $f \in L^2(E)$ satisfies Parseval formula with respect to $\{\varphi_k\}$;

that is, $\|f\|_2 = \left(\sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{1}{2}}$, where $\{c_k\}$ is the sequence of Fourier coefficients of f .

(12%)