Part I: 微積分 (Calculus)

1. (5 pts) For each positive integer n, evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

using the transformation $u = \frac{\pi}{2} - x$.

2. Let Γ be a plane curve defined by the parametric equations

$$x(t) = \ln\sqrt{1-t^2}, \quad y(t) = \arccos t, \quad 0 \le t < 1.$$

(a) (5 pts) Find the slope of Γ when $t = 1/2$.

(b) (5 pta) Find the second derivative $\dfrac{d^2y}{dx^2}$.

3. (a) (5 pts) Evaluate the integral $\displaystyle\int_{-2}^{2} \frac{5}{x^2 - x - 6}\, dx$.

(b) (5 pts) Use the part (a) to find $\displaystyle\lim_{a\to(-2)^+} \int_a^2 \frac{5a+10}{x^2-x-6}\, dx$.

4. (a) (10 pts) Find the Taylor series of $f(x) = \ln(x+1)$ at $x = 2$.

(b) (5 pts) Find the interval of convergence for the series obtained in the part (a).

5. (10 pts) Show that there exists a unique $p \in (0, \frac{\pi}{2})$ so that $\cos p = p$.