# An introduction to Strichartz estimates II

**Lemma 1** (Frequency localisation) For $q \in [2,\infty]$, $r \in [2,\infty)$

$$\|U_0 f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2} \Rightarrow \|Uf\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s}$$

$$\left(s = \frac{n}{2} - \frac{n}{r} - \frac{2}{q}\right)$$

i.e. $H(q,r)$ holds.

Notation • $Uf(t,x) = e^{it\Delta} f(x)$

• Fix non-negative, radial $\chi \in C_c^\infty$ with $\operatorname{supp} \chi \subseteq \{\xi \in \mathbb{R}^n : |\xi| \in [1/2, 2]\}$ "smooth partition of unity"

$$\sum_{k \in \mathbb{Z}} \chi(2^{-k}\xi) = 1 \quad (\forall \xi \neq 0)$$

• $U_0 := UP_0$ where $\widehat{P_0 f}(\xi) = \chi(\xi) \hat{f}(\xi)$

**Goal** $\boxed{|\langle U_0 U_0^* F, G\rangle| \lesssim \|F\|_{L^{q'}_t L^{r'}_x} \|G\|_{L^{q'}_t L^{r'}_x} \quad (H_0(q,r))}$

- By Lemma 1, $H_0(q,r) \Rightarrow H(q,r)$ $\left(q \in [2,\infty], r \in [2,\infty)\right)$
- $U_0 U_0^* F(t,x) \simeq F * K(t,x)$, where

$$K(t,x) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \chi(\xi)^2 \, d\xi$$

**Note** • $\hat{K}_t(\xi) = e^{-it|\xi|^2} \chi(\xi)^2 \quad (K_t(x) = K(t,x))$

• $\sup_{x \in \mathbb{R}^n} |K(t,x)| \lesssim \frac{1}{(1+|t|)^{n/2}} \quad (\forall t \in \mathbb{R}) \quad$ (Lemma 2)

So

$$|\langle U_0 U_0^* F, G\rangle| \lesssim \int_{\mathbb{R}^2} \frac{\|F(\tilde{t},\cdot)\|_{r'} \|G(t,\cdot)\|_{r'}}{(1+|t-\tilde{t}|)^{\frac{n}{2}(1-\frac{2}{r})}} \, dt \, d\tilde{t}$$

$= \frac{2}{q}$ when $\frac{1}{q} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$

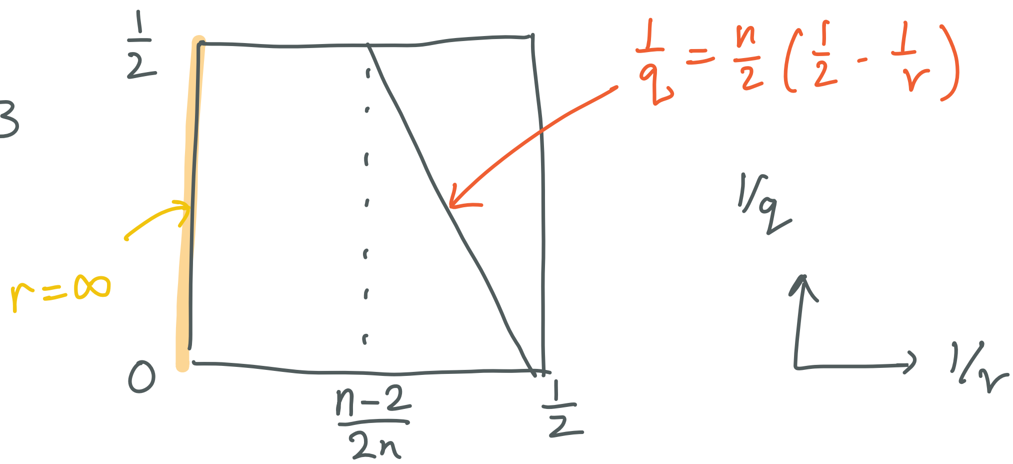
For $q \in [2,\infty]$ and $\frac{1}{q} < \frac{n}{2}\left(\frac{1}{2}-\frac{1}{r}\right)$, Young's convolution gives

$$|\langle U_0 U_0^* F, G\rangle| \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}} \qquad \left(\because \frac{n}{2}\left(1-\frac{2}{r}\right)\frac{q}{2} > 1\right)$$

For $q \in (2,\infty]$ and $\frac{1}{q} = \frac{n}{2}\left(\frac{1}{2}-\frac{1}{r}\right)$, Hardy-Littlewood-Sobolev gives

$$|\langle U_0 U_0^* F, G\rangle| \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}} \qquad \left(\because \frac{n}{2}\left(1-\frac{2}{r}\right) = \frac{2}{q} < 1\right)$$

$n \geq 3$




In conclusion, we have proved $H_0(q,r)$ whenever

$n \geq 1$, $q \in [2,\infty]$, $r \in [2,\infty]$, $\frac{1}{q} \leq \frac{n}{2}\left(\frac{1}{2}-\frac{1}{r}\right)$,

$(q,r) \neq \left(2, \frac{2n}{n-2}\right)$ $(n \geq 2)$

Next we handle the "endpoint case":

Theorem 2 (Keel-Tao, Amer. J. Math 1998)

$H_0\left(2, \frac{2n}{n-2}\right)$ holds for all $n \geq 3$.

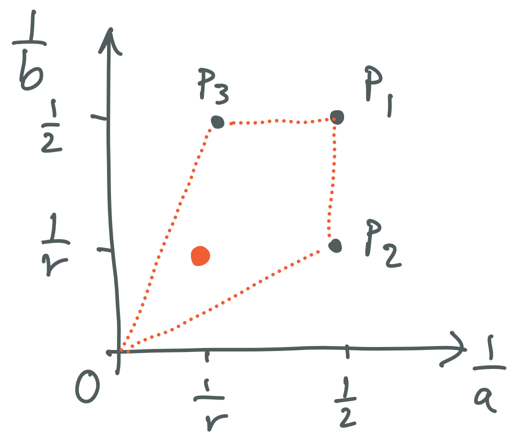
<u>Notation</u> $r = \frac{2n}{n-2}$ , $\sigma = \frac{n}{2}$

$P_1 = (\frac{1}{2}, \frac{1}{2})$

$P_2 = (\frac{1}{2}, \frac{1}{r})$

$P_3 = (\frac{1}{r}, \frac{1}{2})$

$\beta(a,b) = \sigma - 1 - \sigma\left(\frac{1}{a} + \frac{1}{b}\right)$




<u>Lemma 3</u>

$$\left| \int_{2^j \le |t-\tilde{t}| < 2^{j+1}} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)}\, dx\, dt\, d\tilde{x}\, d\tilde{t} \right|$$

$$\lesssim 2^{-j\beta(a,b)} \| F \|_{L^2_t L^{a'}_x} \| G \|_{L^2_t L^{b'}_x}$$

holds whenever $j \in \mathbb{Z}$ and $\left(\frac{1}{a}, \frac{1}{b}\right) \in \operatorname{int}(O P_3 P_1 P_2)$

Let $T_j(F,G) := \int_{2^j \le |t-\tilde{t}| < 2^{j+1}} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)}\, dx\, dt\, d\tilde{x}\, d\tilde{t}$

<u>GOAL</u> $\left| \underbrace{\sum_{j \in \mathbb{Z}} T_j(F,G)}_{\langle U_0 U_0^* F, G \rangle} \right| \lesssim \| F \|_{L^2_t L^{r'}_x} \| G \|_{L^2_t L^{r'}_x}$

Suffices to show $\boxed{\sum_{j \in \mathbb{Z}} |T_j(F,G)| \lesssim \| F \|_{L^2_t L^{r'}_x} \| G \|_{L^2_t L^{r'}_x} \quad (H_0^*(2,r))}$

i.e. $T : L^2_t L^{r'}_x \times L^2_t L^{r'}_x \longrightarrow \ell^1$ is bounded, where

$$T(F,G) = \left(T_j(F,G)\right)_{j\in\mathbb{Z}}$$

Lemma 3 implies

$$|T_j(F,G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$$

$$\forall \left(\tfrac{1}{a},\tfrac{1}{b}\right) \in \mathrm{int}(OP_3P_1P_2)$$

<u>Note</u> $\beta(r,r) = \sigma - 1 - \frac{2\sigma}{r} = 0$ $\left(\sigma = \frac{n}{2},\ r = \frac{2n}{n-2}\right)$ so we cannot directly obtain $H_0^*(2,r)$. It is key that Lemma 3 holds in a neighborhood of $\left(\frac{1}{a},\frac{1}{b}\right) = \left(\frac{1}{r},\frac{1}{r}\right)$

E.g. Suppose $F(\tilde{t},\tilde{x}) = f(\tilde{t})\, 1_{\tilde{E}(\tilde{t})}(x)$ $\left(|\tilde{E}(\tilde{t})| \sim 1,\ \|f\|_2 \sim 1\right)$

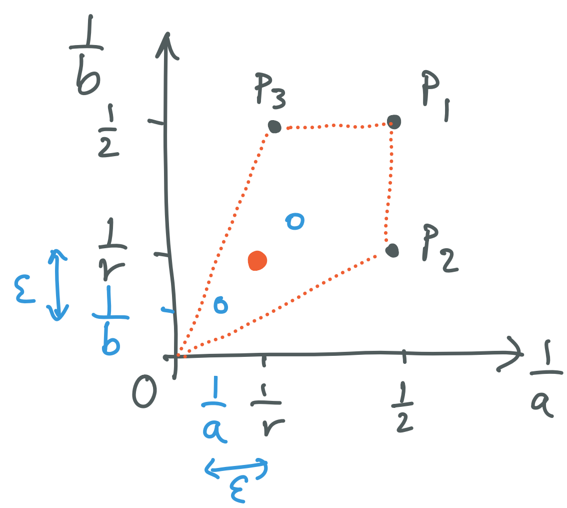
$$G(t,x) = g(t)\, 1_{E(t)}(x) \quad \left(|E(t)| \sim 1,\ \|g\|_2 \sim 1\right)$$

Then Lemma 3 yields

$$|T_j(F,G)| \lesssim 2^{-j\left(\sigma - 1 - \sigma\left(\frac{1}{a}+\frac{1}{b}\right)\right)} \qquad \left(\begin{array}{l}\forall \left(\frac{1}{a},\frac{1}{b}\right) \in \mathrm{int}(OP_3P_1P_2) \\ \forall j \in \mathbb{Z}\end{array}\right)$$

$$= 2^{-j\sigma\left[\left(\frac{1}{r}-\frac{1}{a}\right)+\left(\frac{1}{r}-\frac{1}{b}\right)\right]}$$

$$\left(\begin{array}{l} r = \frac{2\sigma}{\sigma-1} \\ \text{i.e. } \frac{2}{r} = 1 - \frac{1}{\sigma}\end{array}\right)$$



$\underline{j \geq 0}$ take $\begin{cases} \frac{1}{a} = \frac{1}{r} - \varepsilon \\ \frac{1}{b} = \frac{1}{r} - \varepsilon \end{cases}$ ($\varepsilon > 0$ sufficiently small)

to get $|T_j(F,G)| \lesssim 2^{-2\varepsilon\sigma j}$, which sums.

$\underline{j \leq 0}$ take $\begin{cases} \frac{1}{a} = \frac{1}{r} + \varepsilon \\ \frac{1}{b} = \frac{1}{r} + \varepsilon \end{cases}$ ($\varepsilon > 0$ sufficiently small)

to get $|T_j(F,G)| \lesssim 2^{+2\varepsilon\sigma j}$, which sums.

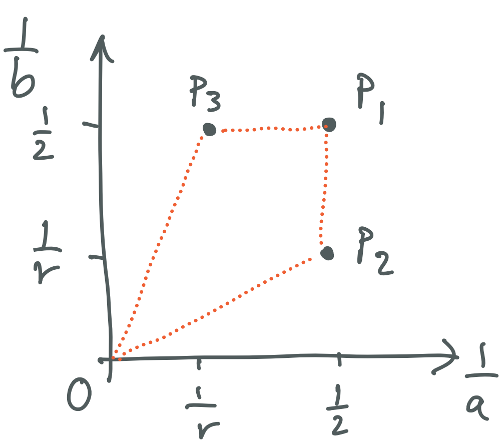
Based on this (and an "atomic decomposition of $L^p$") one can prove $H_0^*(2,r)$ (see §5 of Keel-Tao).

## Proof of Lemma 3

GOAL $|T_j(F,G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$

for all $(\frac{1}{a}, \frac{1}{b}) \in \operatorname{int}(0P_3P_1P_2)$

and all $j \in \mathbb{Z}$.




Let $T_j(F,G) := \int_{2^j \leq |t-\tilde{t}| < 2^{j+1}} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t},\tilde{x}) \overline{G(t,x)}\, dx\, dt\, d\tilde{x}\, d\tilde{t}$

- Reduce to integral over $2^j \leq t-\tilde{t} < 2^{j+1}$ (by symmetry).
- Reduction to "scale 1" in time :

$$\int_{2^j \le t-\hat{t} < 2^{j+1}} K(t-\hat{t}, x-\hat{x}) F(\hat{t},\hat{x}) \overline{G(t,x)}\, dx\,dt\,d\hat{x}\,d\hat{t}$$

$$= 2^{2j} \cdot 2^{2\sigma j} \int_{1 \le t-\tilde{t} < 2} K(2^j(t-\hat{t}), 2^{j/2}(x-\hat{x})) \underbrace{F(2^j\hat{t}, 2^{j/2}\hat{x})}_{F_j(\hat{t},\hat{x})} \overbrace{G(2^j t, 2^{j/2}x)}^{G_j(t,x)}\, dx\,dt\,d\hat{x}\,d\tilde{t}$$

$$= 2^{j(\sigma+2)} \int_{1 \le t-\tilde{t} < 2} K^{(j)}(t-\hat{t}, x-\tilde{x}) F_j(\hat{t},\tilde{x}) G_j(t,x)\, dx\,dt\,d\hat{x}\,d\tilde{t}$$

$$\left( \begin{aligned} K(2^j t, 2^{j/2}x) &= \int_{\mathbb{R}^n} e^{i(2^{j/2}x\cdot\xi - 2^j t|\xi|^2)} \chi(\xi)^2\, d\xi \\ &= 2^{-\sigma j} \underbrace{\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \chi(2^{-j/2}\xi)^2\, d\xi}_{K^{(j)}(t,x)} \end{aligned} \right)$$

Note $\sup_{x\in\mathbb{R}^n} |K^{(j)}(t,x)| \lesssim \frac{1}{|t|^\sigma}$ $(\forall t\neq 0, \forall j\in\mathbb{Z})$

$$\because\ |K^{(j)}(t,x)| = 2^{\sigma j} |K(2^j t, 2^{j/2}x)| \underset{\text{Lemma 2}}{\lesssim} 2^{\sigma j} \frac{1}{|2^j t|^\sigma} = \frac{1}{|t|^\sigma}$$

Claim If $(\frac{1}{a}, \frac{1}{b}) \in \mathrm{int}(O P_3 P_1 P_2)$, then

$$\boxed{\left| \int_{1\le t-\tilde{t}<2} K^{(j)}(t-\hat{t}, x-\tilde{x}) F(\hat{t},\tilde{x}) G(t,x)\, dx\,dt\,d\hat{x}\,d\tilde{t} \right| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}}$$

If true (applied with $F \rightsquigarrow F_j$ and $G \rightsquigarrow G_j$)

$$\left| \int_{2^j \le t-\hat{t} < 2^{j+1}} K(t-\hat{t}, x-\hat{x}) F(\hat{t},\hat{x}) \overline{G(t,x)}\, dx\,dt\,d\hat{x}\,d\hat{t} \right| \lesssim 2^{j(\sigma+2)} \|F_j\|_{L^2_t L^{a'}_x} \|G_j\|_{L^2_t L^{b'}_x}$$

$$= \frac{2^{j(\sigma+2)}}{2^{\frac{j}{2}} 2^{j\frac{\sigma}{a'}} \cdot 2^{\frac{j}{2}} 2^{j\frac{\sigma}{b'}}} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$$

$$= 2^{-j\beta(a,b)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$$

Proof of claim • Reduce to $O(1)$-support (temporal)

i.e. it suffices to prove

$$\left|\int_{1\le t-\tilde t<2} K^{(j)}(t-\tilde t,\, x-\tilde x)\, F(\tilde t,\tilde x)\, G(t,x)\, dx\, dt\, d\tilde x\, d\tilde t\right| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$$

where $I, \tilde I$ are intervals, $\begin{cases} \tilde t \notin \tilde I \Rightarrow F(\tilde t,\cdot) \equiv 0 \\ t \notin I \Rightarrow G(t,\cdot)\equiv 0 \end{cases}$

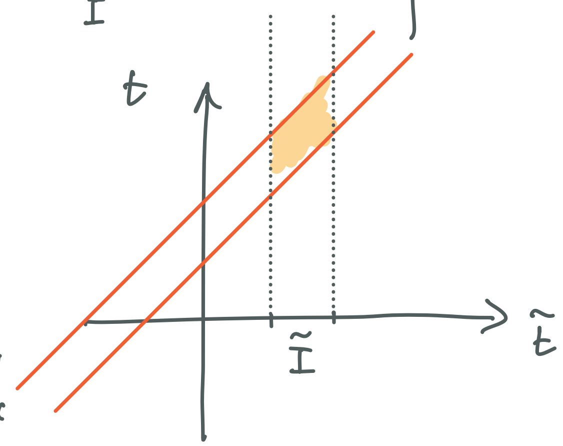
length$(I)$ = length$(\tilde I)$ $= 1$

Write $F(\tilde t,\tilde x) = \sum_{\tilde I} \underbrace{F(\tilde t,\tilde x)\,1_{\tilde I}(\tilde t)}_{F_{\tilde I}(\tilde t,\tilde x)}$, $\quad G(t,x) = \sum_I \underbrace{G(t,x)\,1_I(t)}_{G_I(t,x)}$

Then $\left|\int_{1\le t-\tilde t<2} K^{(j)}(t-\tilde t,\, x-\tilde x)\, F(\tilde t,\tilde x)\, G(t,x)\, dx\, dt\, d\tilde x\, d\tilde t\right|$

$$\le \sum_{\tilde I} \sum_{I} \left|\int_{1\le t-\tilde t<2} K^{(j)}(t-\tilde t,\, x-\tilde x)\, F_{\tilde I}(\tilde t,\tilde x)\, G_I(t,x)\, dx\, dt\, d\tilde x\, d\tilde t\right|$$

$I \in \mathcal{C}(\tilde I)$, $\#\mathcal{C}(\tilde I) \sim 1$




$$\lesssim \sum_{\tilde I} \sum_{I\in \mathcal{C}(\tilde I)} \|F_{\tilde I}\|_{L^2_t L^{a'}_x} \|G_I\|_{L^2_t L^{b'}_x}$$

$$\lesssim \Big(\sum_{\tilde I} \|F_{\tilde I}\|^2_{L^2_t L^{a'}_x}\Big)^{\frac12} \Big(\sum_{\tilde I} \|G_{\tilde I}\|^2_{L^2_t L^{b'}_x}\Big)^{1/2} \quad \text{(Cauchy-Schwarz)}$$

$$= \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}.$$

$\left(\frac{1}{a},\frac{1}{b}\right)=(0,0)$ By the dispersive estimate

$$\left|\int_{1\le t-\tilde{t}<2} K^{(j)}(t-\tilde{t},\ x-\tilde{x})\, F(\tilde{t},\tilde{x})\, G(t,x)\, dx\,dt\,d\tilde{x}\,d\tilde{t}\right|$$

$$\lesssim \int_{1\le t-\tilde{t}<2} \frac{1}{|t-\tilde{t}|^{\sigma}} |F(\tilde{t},\tilde{x})|\ |G(t,x)|\ dx\,dt\,d\tilde{x}\,d\tilde{t}$$

$$\sim \int_{1\le t-\tilde{t}<2} |F(\tilde{t},\tilde{x})|\,|G(t,x)|\,1_{\tilde{I}}(\tilde{t})\,1_{I}(t)\,dx\,d\tilde{x}\,dt\,d\tilde{t}$$

$$\le \|F1_{\tilde{I}}\|_{L^1_tL^1_x}\,\|G1_I\|_{L^1_tL^1_x} \underset{\uparrow\ \text{Hölder}}{\lesssim} \|F1_{\tilde{I}}\|_{L^2_tL^1_x}\|G1_I\|_{L^2_tL^1_x}$$

$\left(\frac{1}{a},\frac{1}{b}\right)\in[P_1,P_3)$ $\left(\text{i.e. } b=2,\ \ \frac{1}{a}\in\left(\frac{1}{r},\frac{1}{2}\right]\right)$

$F1_{J(t)}=:H^{(t)}$

$J(t)=(t-2,t-1]$

Length $(I)=1$.

$$\left|\int_{1\le t-\tilde{t}<2} K^{(j)}(t-\tilde{t},\ x-\tilde{x})\, F(\tilde{t},\tilde{x})\, \overline{G(t,x)}\, dx\,dt\,d\tilde{x}\,d\tilde{t}\right|$$

$$=\left|\int_I\int\iint K^{(j)}(t-\tilde{t},x-\tilde{x})H^{(t)}(\tilde{t},\tilde{x})\,d\tilde{t}\,d\tilde{x}\ \overline{G(t,x)}\ dx\ dt\right|$$

$$=\left|\int_I\int\iint\left(\int e^{i((x-\tilde{x})\cdot\xi-(t-\tilde{t})|\xi|^2)}\chi(2^{-j/2}\xi)^2d\xi\ H^{(t)}(\tilde{t},\tilde{x})\,d\tilde{t}\,d\tilde{x}\right)\overline{G(t,x)}\ dx\ dt\right|$$

$$=\left|\int_I\int\iint e^{-i(\tilde{x}\cdot\xi-\tilde{t}|\xi|^2)}H^{(t)}(\tilde{t},\tilde{x})\,d\tilde{t}\,d\tilde{x}\int e^{i(x\cdot\xi-t|\xi|^2)}\overline{G(t,x)}dx\ \chi(2^{-j/2}\xi)^2d\xi\,dt\right|$$

$$\le \sup_{t\in I} N_1(t)^{\frac{1}{2}}\int_I N_2(t)^{\frac{1}{2}}\,dt$$

$$N_1(t) := \int \left| \iint e^{-i(\tilde{x}\cdot\xi - \tilde{t}|\xi|^2)} H^{(t)}(\tilde{t},\tilde{x})\, d\tilde{t}\, d\tilde{x} \right|^2 \chi(2^{-j/2}\xi)^2\, d\xi$$

$$N_2(t) := \int \left| \int e^{i(x\cdot\xi - t|\xi|^2)} \overline{G(t,x)}\, dx \right|^2 \underbrace{\chi(2^{-j/2}\xi)^2}_{\lesssim 1}\, d\xi$$

Now
$$N_2(t) \lesssim \int \left| \int e^{-ix\cdot\xi} e^{+it|\xi|^2} G(t,x)\, dx \right|^2 d\xi$$
$$\simeq \int |G(t,x)|^2\, dx \qquad \text{(Plancherel)}$$

So
$$\int_I N_2(t)^{\frac{1}{2}}\, dt \lesssim \|G\|_{L^1_t L^2_x} \underset{\text{Hölder}}{\lesssim} \|G\|_{L^2_t L^2_x}.$$

Also,
$$N_1(t) = \int \left| \iint e^{-i(\tilde{x}\cdot\xi - \tilde{t}|\xi|^2)} H^{(t)}(\tilde{t},\tilde{x})\, d\tilde{t}\, d\tilde{x} \right|^2 \chi(2^{-j/2}\xi)^2\, d\xi$$

$$\underset{\eta = 2^{-j/2}\xi}{=} 2^{\sigma j} \int \Bigg| \underbrace{\iint e^{-i(\tilde{x}\cdot\xi - \tilde{t}|\xi|^2)} \underbrace{H^{(t)}(2^{-j}\tilde{t}, 2^{-j/2}\tilde{x})}_{H^{(t)}_{-j}(\tilde{t},\tilde{x})} \frac{d\tilde{t}\, d\tilde{x}}{2^j 2^{j\sigma}}}_{\widehat{H^{(t)}_{-j}}(-|\xi|^2,\xi)} \Bigg|^2 \chi(\xi)^2\, d\xi$$

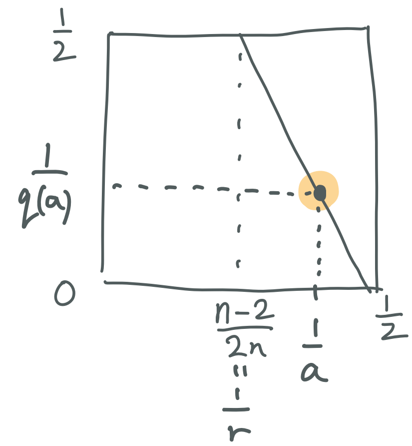
$$= \frac{1}{2^{j(\sigma+2)}} \int \left| \widehat{U_0^* H^{(t)}_{-j}}(\xi) \right|^2 d\xi$$

$$\simeq \frac{1}{2^{j(\sigma+2)}} \| U_0^* H^{(t)}_{-j} \|_2^2$$

$$= \frac{1}{2^{j(\sigma+2)}} \langle U_0 U_0^* H^{(t)}_{-j}, H^{(t)}_{-j} \rangle$$

$$\le \frac{1}{2^{j(\sigma+2)}} \left\| U_0 U_0^* H^{(t)}_{-j} \right\|_{L^{q(a)}_t L^{a}_x} \left\| H^{(t)}_{-j} \right\|_{L^{q(a)'}_t L^{a'}_x}$$

$n \ge 3$




$\frac{1}{q(a)} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{a}\right)$

$\left( q(a) > 2, \text{ so } q(a)' < 2 \right)$

Non-endpoint Strichartz!

$$\lesssim \frac{1}{2^{j(\sigma+2)}} \left\| H^{(t)}_{-j} \right\|^2_{L^{q(a)'}_t L^{a'}_x} \qquad \left(\text{using here that } \frac{1}{a} > \frac{1}{r}\right)$$

Now
$$\left\| H^{(t)}_{-j} \right\|^2_{L^{q(a)'}_t L^{a'}_x} = 2^{j\left(\frac{2}{q(a)'} + \frac{2\sigma}{a'}\right)} \left\| H^{(t)} \right\|^2_{L^{q(a)'}_t L^{a'}_x}$$
$$= 2^{j\left(\frac{2}{q(a)'} + \frac{2\sigma}{a'}\right)} \left( \int_{[t-2,t-1]} \left( \int |F(\tilde{t},\tilde{x})|^{a'} \, d\tilde{x} \right)^{\frac{q(a)'}{a'}} d\tilde{t} \right)^{\frac{2}{q(a)'}}$$
$$\underset{\text{Hölder}}{\lesssim} 2^{j\left(\frac{2}{q(a)'} + \frac{2\sigma}{a'}\right)} \left\| F \right\|^2_{L^2_t L^{a'}_x}$$

Hence $N_1(t) \lesssim \| F \|^2_{L^2_t L^{a'}_x}$.

Whence
$$\left| \iint_{1 \le t - \tilde{t} < 2} K^{(j)}(t-\tilde{t}, x-\tilde{x}) F(\tilde{t},\tilde{x}) \overline{G(t,x)} \, dx \, dt \, d\tilde{x} \, d\tilde{t} \right| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^2_x} \quad \blacksquare$$

Proof of $H_0^*(2,r)$ using bilinear interpolation (see §6 of Keel-Tao).

Lemma 4 (see Exercise 5, p76 of Interpolation Spaces: An Introduction by Bergh-Löfström)

For appropriate Banach spaces $A_0, A_1, B_0, B_1, C_0, C_1$, if $T$ is bilinear and
$$T: \begin{cases} A_0 \times B_0 \to C_0 \\ A_0 \times B_1 \to C_1 \\ A_1 \times B_0 \to C_1 \end{cases} \quad \text{bounded}$$

then $T: (A_0, A_1)_{\theta_0, p_0} \times (B_0, B_1)_{\theta_1, p_1} \to (C_0, C_1)_{\theta, 1}$

where $\begin{cases} p_0, p_1 \in [1,\infty] \text{ satisfy } \frac{1}{p_0} + \frac{1}{p_1} \geq 1 \\ \theta_0, \theta_1, \theta \in (0,1) \text{ satisfy } \theta = \theta_0 + \theta_1. \end{cases}$

Note • For definition of real interpolation spaces $(A_0, A_1)_{\theta, p}$ see Bergh-Löfström.

• For Lorentz spaces (see Theorem 5.3.1 of Bergh-Löfström)

$$\left(L^{q_0, p_0}, L^{q_1, p_1}\right)_{\theta, p} = L^{q_\theta, p} \quad \left(\begin{array}{c} p, p_0, p_1, q_0, q_1 \in [1,\infty] \\ q_0 \neq q_1 \\ \theta \in (0,1),\ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \end{array}\right)$$

Also $\left(L^{q, p_0}, L^{q, p_1}\right)_{\theta, p_\theta} = L^{q, p_\theta} \quad \left(\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)$

Recall • $L^{q,q} = L^q$

• $L^{q, p_0} \subseteq L^{q, p_1}$ if $p_1 \geq p_0$.

• For mixed-norms, care is needed!

For appropriate $A_0, A_1$, we have

$$(L^{q_0}(A_0), L^{q_1}(A_1))_{\theta, q_\theta} = L^{q_\theta}((A_0, A_1)_{\theta, q_\theta}) \qquad \left( \begin{matrix} q_0, q_1 \in [1,\infty] \\ \theta \in (0,1) \end{matrix} \right)$$

(Lions - Peetre, Inst. Hautes Etudes Sci. Publ. Math. 1964)

We will use the special case ($p_0, p_1 \in [1,\infty]$, $p_0 \neq p_1$, $\theta \in (0,1)$)

$$(L^2_t L^{p_0}_x, L^2_t L^{p_1}_x)_{\theta, 2} = L^2_t((L^{p_0}_x, L^{p_1}_x)_{\theta, 2})$$
$$= L^2_t L^{p_\theta, 2}_x$$

Question What about $(L^{q_0}(A_0), L^{q_1}(A_1))_{\theta, p}$ for other $p$?

It turns out that there is no reasonable extension of the Lions-Peetre formula! (Cwikel, Proc. Amer. Math. Soc. 1974).

(If $A_0 = A_1 = A$, then $(L^{q_0}(A), L^{q_1}(A))_{\theta, p} = L^{q_\theta, p}(A)$.)

Notation
$$\|(a_j)\|_{\ell^1_\beta} := \sum_{j \in \mathbb{Z}} 2^{j\beta} |a_j| \qquad (\beta \in \mathbb{R})$$
$$\|(a_j)\|_{\ell^\infty_\beta} := \sup_{j \in \mathbb{Z}} 2^{j\beta} |a_j|$$

Then
$$(\ell^\infty_{\beta_0}, \ell^\infty_{\beta_1})_{\theta, 1} = \ell^1_\beta \qquad \begin{matrix} \beta = (1-\theta)\beta_0 + \theta\beta_1 \\ \beta_0 \neq \beta_1, \ \theta \in (0,1) \end{matrix}$$

(see Theorem 5.6.1 of Bergh-Löfström)

Recall Lemma 3 implies

$$2^{j\beta(a,b)} |T_j(F,G)| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x} \quad \left(\forall \left(\tfrac{1}{a},\tfrac{1}{b}\right) \in \text{int}(0P_3P_1P_2)\right)$$

i.e. $T : L^2_t L^{a'}_x \times L^2_t L^{b'}_x \to \ell^\infty_{\beta(a,b)}$

GOAL $$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \lesssim \|F\|_{L^2_t L^{r'}_x} \|G\|_{L^2_t L^{r'}_x} \quad \left(r=\frac{2n}{n-2},\ n\geq 3\right)$$

i.e. $T : L^2_t L^{r'}_x \times L^2_t L^{r'}_x \longrightarrow \ell^1$