An introduction to Strichartz estimates III

Theorem 2 (Keel-Tao, Amer. J. Math 1998)

$\boxed{H_0\left(2, \frac{2n}{n-2}\right) \text{ holds for all } n \geq 3.}$

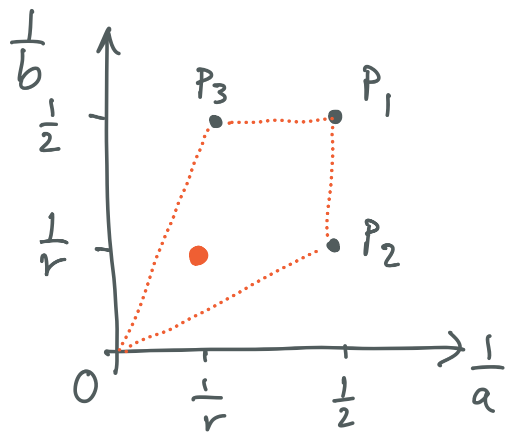
Notation $r = \frac{2n}{n-2}$, $\sigma = \frac{n}{2}$

$P_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$

$P_2 = \left(\frac{1}{2}, \frac{1}{r}\right)$

$P_3 = \left(\frac{1}{r}, \frac{1}{2}\right)$

$\beta(a,b) = \sigma - 1 - \sigma\left(\frac{1}{a} + \frac{1}{b}\right)$



Lemma 3 (Key lemma)

For $j \in \mathbb{Z}$ and $\left(\frac{1}{a}, \frac{1}{b}\right) \in \operatorname{int}(OP_3P_1P_2)$,

$$\left| \int_{2^j \leq |t-\tilde{t}| < 2^{j+1}} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)} \, dx\, dt\, d\tilde{x}\, d\tilde{t} \right|$$

$$\lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$$

Let $T_j(F,G) := \int_{2^j \leq |t-\tilde{t}| < 2^{j+1}} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)} \, dx\, dt\, d\tilde{x}\, d\tilde{t}$

For Theorem 2, it suffices to prove:

$$\sum_{j \in \mathbb{Z}} |T_j(F,G)| \lesssim \|F\|_{L^2_t L^{r'}_x} \|G\|_{L^2_t L^{r'}_x} \qquad \left(\begin{array}{c} r = \frac{2n}{n-2} \\ n \geq 3 \end{array} \right)$$

i.e. $\quad \|T(F,G)\|_{\ell^1} \lesssim \|F\|_{L^2_t L^{r'}_x} \|G\|_{L^2_t L^{r'}_x}$

where $\quad T(F,G) := (T_j(F,G))_{j \in \mathbb{Z}}$.

Lemma 4

For appropriate Banach spaces $A_0, A_1, B_0, B_1, C_0, C_1$, if T is bilinear, then

$$T: \begin{cases} A_0 \times B_0 \to C_0 \\ A_0 \times B_1 \to C_1 \\ A_1 \times B_0 \to C_1 \end{cases} \Rightarrow T: (A_0, A_1)_{\theta_0, p_0} \times (B_0, B_1)_{\theta_1, p_1} \to (C_0, C_1)_{\theta, 1}$$

where $\begin{cases} p_0, p_1 \in [1, \infty] \text{ satisfy } \frac{1}{p_0} + \frac{1}{p_1} \geq 1 \\ \theta_0, \theta_1, \theta \in (0,1) \text{ satisfy } \theta = \theta_0 + \theta_1. \end{cases}$

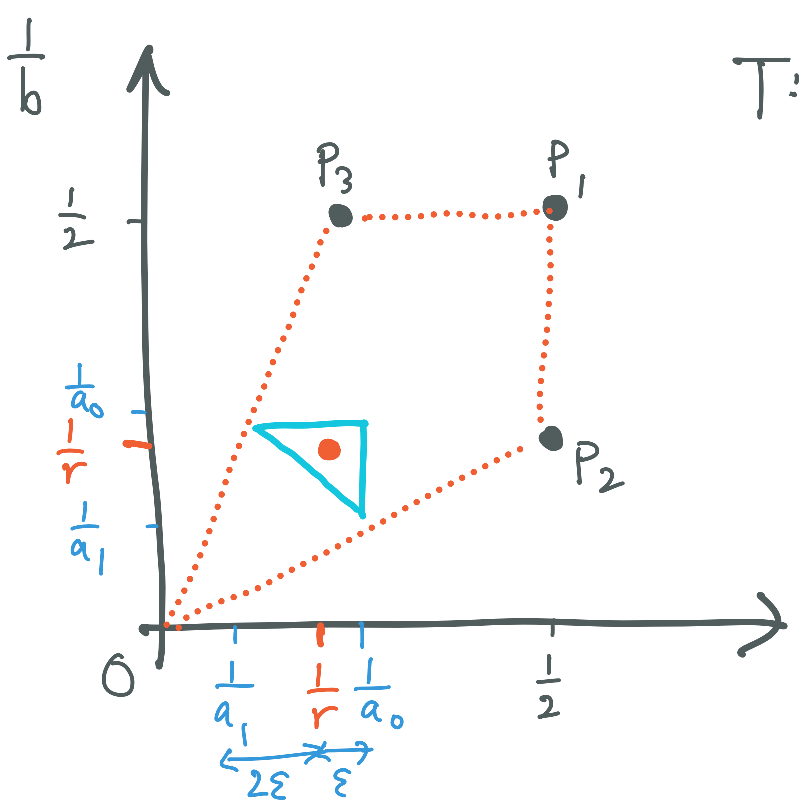
Recall: $\|(a_j)\|_{\ell^1_\beta} := \sum_{j \in \mathbb{Z}} 2^{j\beta} |a_j|$

$\|(a_j)\|_{\ell^\infty_\beta} := \sup_{j \in \mathbb{Z}} 2^{j\beta} |a_j|$

- We use $(\ell^\infty_{\beta_0}, \ell^\infty_{\beta_1})_{\theta, 1} = \ell^1_\beta$

- Lemma 3 implies $2^{j\beta(a,b)} |T_j(F,G)| \lesssim \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$ $\left(\forall \left(\frac{1}{a}, \frac{1}{b}\right) \in \mathrm{int}(OP_3P_1P_2)\right)$

  i.e. $T: L^2_t L^{a'}_x \times L^2_t L^{b'}_x \to \ell^\infty_{\beta(a,b)}$



$$T: \begin{cases} A_0 \times A_0 \to C_0 \\ A_0 \times A_1 \to C_1 \\ A_1 \times A_0 \to C_1 \end{cases}$$

$A_0 = L^2_t L^{a_0'}_x \quad (= B_0)$

$A_1 = L^2_t L^{a_1'}_x \quad (= B_1)$

$\frac{1}{a_0} = \frac{1}{r} + \varepsilon$

$\frac{1}{a_1} = \frac{1}{r} - 2\varepsilon$

($\varepsilon > 0$: sufficiently small)

$C_0 = \ell^\infty_{\beta(a_0, a_0)}$

$C_1 = \ell^\infty_{\beta(a_0, a_1)}$

GOAL $T: L^2_t L^{r'}_x \times L^2_t L^{r'}_x \to \ell^1$

Note • $(C_0, C_1)_{\frac{2}{3},1} = \left(\ell^\infty_{\beta(a_0,a_0)}, \ell^\infty_{\beta(a_0,a_1)}\right)$

$$= \ell^1_0$$

$\left((\ell^\infty_{\beta_0}, \ell^\infty_{\beta_1})_{\theta,1} = \ell^1_\beta \right.$, $\left. \beta = (1-\theta)\beta_0 + \theta\beta_1 \right)$

• $(A_0, A_1)_{\frac{1}{3},2} = \left(L^2_t L^{a_0'}_x, L^2_t L^{a_1'}_x\right)_{\frac{1}{3},2}$

$$= L^2_t L^{r',2}_x$$

Lions-Peetre formula $\left(L^2_t L^{p_0}_x, L^2_t L^{p_1}_x\right)_{\theta,2} = L^2_t L^{p_\theta,2}_x$ $\quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

Lemma 4 gives

$$\underbrace{\sum_{j\in\mathbb{Z}} |T_j(F,G)|}_{\|T(F,G)\|_{\ell^1}} \lesssim \|F\|_{L^2_t L^{r',2}_x} \|G\|_{L^2_t L^{r',2}_x}$$

$$\lesssim \|F\|_{L^2_t L^{r'}_x} \|G\|_{L^2_t L^{r'}_x}$$

since $r = \frac{2n}{n-2} > 2$, so $r' < 2$ which means $L^{r'} = L^{r',r'} \subseteq L^{r',2}$

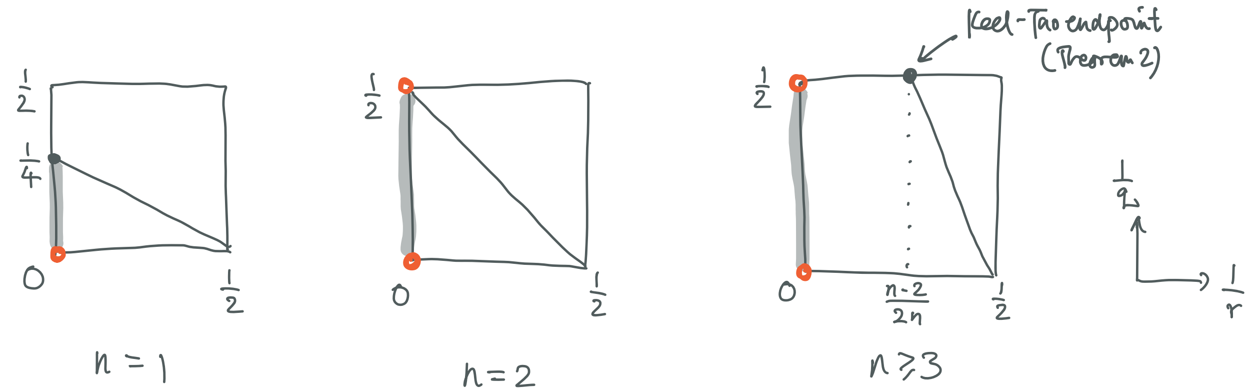
∎

Consider $H(q,\infty)$: $\|e^{it\Delta}f\|_{L^q_t L^\infty_x} \lesssim \|f\|_{\dot{H}^s}$ $(s=\frac{n}{2}-\frac{2}{q})$

Theorem 1 when $r=\infty$

Let $n\geq 1$, $s=\frac{n}{2}-\frac{2}{q}$.

- If $n=1$, $q\in[4,\infty)$ then $H(q,\infty)$ holds.
- If $n\geq 2$, $q\in(2,\infty)$ then $H(q,\infty)$ holds.



Remarks • $H(2,\infty)$ fails for $n\geq 2$.
($n=2$: Montgomery-Smith, $n\geq 3$: Guo-Li-Nakanishi-Yan)

- When $n=2$, $\|e^{it\Delta}f\|_{L^2_t BMO_x} \lesssim \|f\|_{L^2}$ fails (Montgomery-Smith)
  $= \dot{H}^s$ $(s=0$ in this case$)$
- When $n\geq 3$, $\|e^{it\Delta}f\|_{L^2_t BMO_x} \lesssim \|f\|_{\dot{H}^s}$ holds

Follows because $\|e^{it\Delta}P_0 f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2}$ holds for $n\geq 3$ (as we proved in Lecture I), and $L^\infty \to BMO$ Littlewood-Paley estimate. (i.e. slightly modify Lemma 1)

Proof of $H(q,\infty)$

Goal $\| \underbrace{e^{it\Delta} D^{-s}}_{T} f \|_{L^q_t L^\infty_x} \lesssim \|f\|_{L^2}$ $\quad (D = (-\Delta)^{\frac{1}{2}})$

Equivalently $\|TT^* F\|_{L^q_t L^\infty_x} \lesssim \|F\|_{L^{q'}_t L^1_x}$ where $TT^*F = F * K_s$ and

$$K_s(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \frac{1}{|\xi|^{2s}} \, d\xi$$

Alternatively $TT^*F(t,x) = \int e^{i((x-\tilde{x})\cdot\xi - (t-\tilde{t})|\xi|^2)} \frac{1}{|\xi|^{2s}} F(\tilde{t},\tilde{x}) \, d\tilde{x} \, d\xi \, d\tilde{t}$

$$= \int \left(\int e^{i(x\cdot\xi - (t-\tilde{t})|\xi|^2)} \frac{\widehat{F(\tilde{t},\cdot)}(\xi)}{|\xi|^{2s}} \, d\xi \right) d\tilde{t}$$

So

$$\|TT^*F\|_{L^q_t L^\infty_x} \leq \left\| \int_{\mathbb{R}} \sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi - (t-\tilde{t})|\xi|^2)} \frac{\widehat{F(\tilde{t},\cdot)}(\xi)}{|\xi|^{2s}} \, d\xi \right| d\tilde{t} \right\|_{L^q_t}$$

$$\lesssim \left\| \int_{\mathbb{R}} \frac{\|F(\tilde{t},\cdot)\|_{L^1}}{|t-\tilde{t}|^{2/q}} \, d\tilde{t} \right\|_{L^q_t} \lesssim \|F\|_{L^{q'}_t L^1_x}$$

Hardy-Littlewood-Sobolev
(this is valid for $q \in (2,\infty)$)

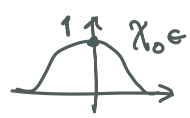
For this we need the "dispersive estimate"

$$\sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \frac{\hat{f}(\xi)}{|\xi|^{\underbrace{2s}_{n-\frac{4}{q}}}} \, d\xi \right| \lesssim \frac{\|f\|_{L^1_x}}{|t|^{2/q}}$$

Which holds for $q \geq \frac{4}{n}$.

But this follows from

$$\sup_{x\in\mathbb{R}^n} \left| \int e^{i(x\cdot\xi - t|\xi|^2)} \frac{\chi_0(\xi)}{|\xi|^{2s}} \, d\xi \right| \lesssim \frac{1}{|t|^{2/q}}$$



$\chi_0 \in C^\infty_c$

which can be proved (using Lemma 2 and a dyadic decomposition in ξ) for $q \geq \frac{4}{n}$ (see Lemma 2.2 in Guo-Li-Nakanishi-Yan). ■

Inhomogeneous Strichartz estimates

Recall $H(q,r)$ when $s=0$: $\|Uf\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}$ $\left(\frac{1}{q} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right)\right)$

Here $Uf(t,x) = e^{it\Delta} f(x)$.

Note $U: L^2 \to L^q_t L^r_x \iff U^*: L^{q'}_t L^{r'}_x \to L^2$
$\iff UU^*: L^{q'}_t L^{r'}_x \to L^q_t L^r_x$

More broadly, may consider

$$\boxed{\|UU^*F\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \qquad (I(\tilde{q},\tilde{r};q,r))}$$

Note • $H(q,r) \iff I(q,r;q,r)$
• $H(q,r)$ and $H(\tilde{q},\tilde{r}) \implies I(\tilde{q},\tilde{r};q,r)$

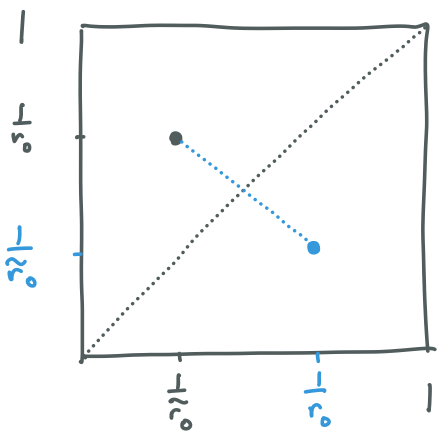
$$\left(\begin{aligned} \because\ |\langle UU^*F, G\rangle| = |\langle U^*F, U^*G\rangle| &\leq \|U^*F\|_2 \|U^*G\|_2 \\ &\lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \|G\|_{L^{q'}_t L^{r'}_x} \end{aligned}\right)$$

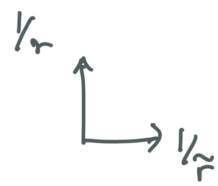
• $I(\tilde{q},\tilde{r};q,r)$ holds $\implies \frac{1}{\tilde{q}'} - \frac{1}{q} + \frac{n}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right) = 1$

by a rescaling argument $(F \rightsquigarrow F(R^2t, Rx))$

Scaling condition

Consider $\left(\frac{1}{\tilde{r}}, \frac{1}{r}\right)$-plane : (Last lecture : $\frac{1}{b}$ ↑ → $1/a$)





Suppose we know $I(\tilde{q}, \tilde{r}_0 ; q, r_0)$ with $q \in \Lambda$ (then $\tilde{q} \in \tilde{\Lambda}$ determined by scaling)

i.e. $UU^* : L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} \to L_t^{q} L_x^{r_0}$, so

$UU^* : L_t^{q'} L_x^{r_0'} \to L_t^{\tilde{q}} L_x^{\tilde{r}_0}$

i.e. we know $I(q, r_0 ; \tilde{q}, \tilde{r}_0)$ with $\tilde{q} \in \tilde{\Lambda}$ (then $q \in \Lambda$ determined by scaling)

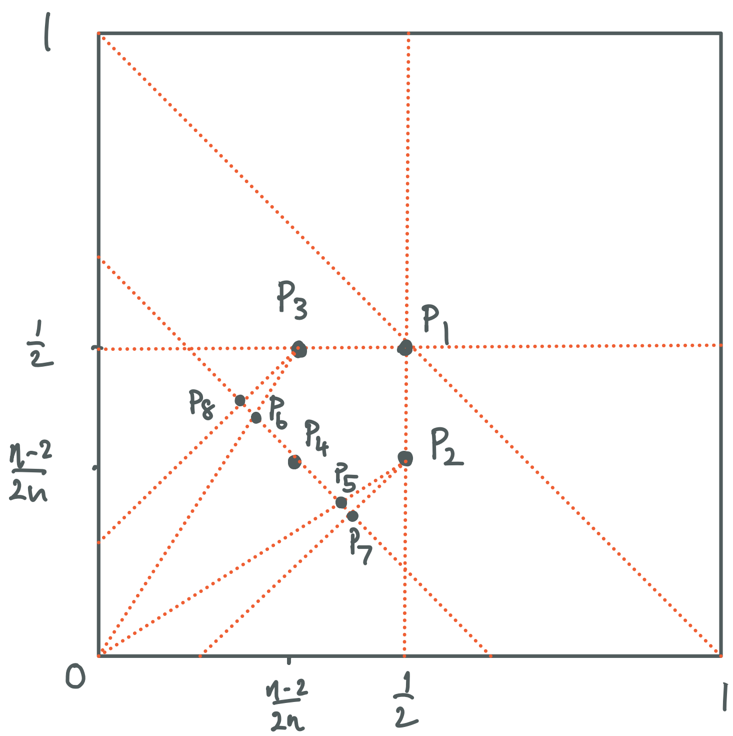
So we may focus on the upper half $\frac{1}{r} \geq \frac{1}{\tilde{r}}$.

- Scaling condition $\frac{1}{\tilde{q}'} - \frac{1}{q} + \frac{n}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right) = 1$

  is equivalent $\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{n}{2}\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right)$

holds when $\frac{1}{q} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$ and $\frac{1}{\tilde{q}} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right)$ (← scaling for $H(q,r)$ and $H(\tilde{q}, \tilde{r})$)

but other cases are possible!



$n \geq 3$

$$\|UU^*F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

$P_1 = (\frac{1}{2}, \frac{1}{2})$ $P_2 = (\frac{1}{2}, \frac{n-2}{2n})$ $P_3 = (\frac{n-2}{2n}, \frac{1}{2})$ $P_4 = (\frac{n-2}{2n}, \frac{n-2}{2n})$

$P_5 = \left(\frac{n-2}{2(n-1)}, \frac{(n-2)^2}{2n(n-1)}\right)$, $P_6 = \left(\frac{(n-2)^2}{2n(n-1)}, \frac{n-2}{2(n-1)}\right)$,

$P_7 = \left(\frac{n-1}{2n}, \frac{n-3}{2n}\right)$, $P_8 = \left(\frac{n-3}{2n}, \frac{n-1}{2n}\right)$.

Theorem 2 (Keel-Tao endpoint $H(2, \frac{2n}{n-2})$ for $n \geq 3$) implies $I(2, \frac{2n}{n-2}; 2, \frac{2n}{n-2})$ i.e. at P_4 with $q = 2$ (hence $\tilde{q} = 2$).

Here we prove:

Theorem 3 (Vilela (Trans. Amer. Math. Soc. 2006), Foschi (J. Hyperbolic Differ. Equ. 2005))

Let $n \geq 3$. If $q \in [\frac{2n}{n+2}, \frac{2n}{n-2}]$, then $I(q', \frac{2n}{n-2}; q, \frac{2n}{n-2})$ holds; i.e. at P_4 with $q \in [\frac{2n}{n+2}, \frac{2n}{n-2}]$.

Remarks

- $UU^* F(t,x) = F * K(t,x)$ where $K(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} d\xi$.

So we have

$$UU^* F(t,x) \simeq \int \left(\int e^{i(x\cdot\xi - (t-\tilde{t})|\xi|^2)} \widehat{F(\tilde{t},\cdot)}(\xi)\, d\xi \right) d\tilde{t}$$

$$\simeq \int e^{i(t-\tilde{t})\Delta} F(\tilde{t},\cdot)(x)\, d\tilde{t}$$

i.e.

$$I(\tilde{q},\tilde{r}; q,r) \iff \left\| \int_{\mathbb{R}} e^{i(t-\tilde{t})\Delta} F(\tilde{t},\cdot)(x) d\tilde{t} \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

- Formally, solution of inhomogeneous Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u = F(t,x) & (t,x) \in \mathbb{R}\times\mathbb{R}^n \\ u(0,x) = f(x) \end{cases}$$

can be written (Duhamel principle)

$$u(t,x) = \underbrace{e^{it\Delta} f(x)}_{\text{"controlled by } H(q,r)\text{"}} - \underbrace{i\int_0^t e^{i(t-\tilde{t})\Delta} F(\tilde{t},\cdot)(x)\, d\tilde{t}}_{\text{"controlled by } I(\tilde{q},\tilde{r};q,r)\text{"}}$$

Note $\left\| \int_{-\infty}^t e^{i(t-\tilde{t})\Delta} F(\tilde{t},\cdot) d\tilde{t} \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$

implies $\left\| \int_0^t e^{i(t-\tilde{t})\Delta} F(\tilde{t},\cdot) d\tilde{t} \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$

and $\left\| \int_{-\infty}^{\infty} e^{i(t-\tilde{t})\Delta} F(\tilde{t},\cdot) d\tilde{t} \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$

(see p2124 of Vilela for details).

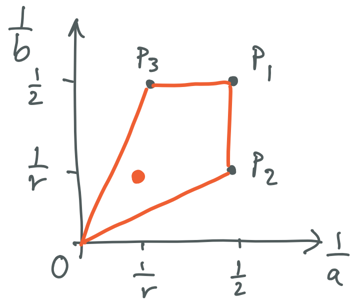
- See e.g. Foschi and Vilela for further discussion of inhomogeneous Strichartz estimates, and results more general than Theorem 3.
- $I(\tilde{q}, \tilde{r}; q, r)$ fails for $(\frac{1}{\tilde{r}}, \frac{1}{r})$ outside $P_1 P_2 P_7 P_8 P_3$ for any q.
- Further progress since Foschi/Vilela (e.g. Koh (2011), Koh–Seo (2016) for progress at P_5, P_6 ; Taggart (2010) ; ...)
- The problem of fully characterizing $(\tilde{q}, \tilde{r}, q, r)$ for which $I(\tilde{q}, \tilde{r}; q, r)$ holds is still open.

Proof of Theorem 3

Let $n \geq 3$. If $q \in [\frac{2n}{n+2}, \frac{2n}{n-2}]$, then $I(q', \frac{2n}{n-2}; q, \frac{2n}{n-2})$

We follow the argument for Theorem 2.

Notation $r = \frac{2n}{n-2}$, $\sigma = \frac{n}{2}$, $\beta(a,b) = \sigma - 1 - \sigma(\frac{1}{a} + \frac{1}{b})$



Lemma 5 (Key lemma)

For $j \in \mathbb{Z}$,

$$\left| \int_{2^j \leq |t - \tilde{t}| < 2^{j+1}} K(t - \tilde{t}, x - \tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)} \, dx \, dt \, d\tilde{x} \, d\tilde{t} \right|$$

$$\lesssim 2^{-j\beta(a,b)} \| F \|_{L_t^{q} L_x^{a'}} \| G \|_{L_t^{q'} L_x^{b'}}$$

whenever

$$\begin{cases} (\frac{1}{a}, \frac{1}{b}) \in \operatorname{int}(0 P_3 P_1) & \text{with } \frac{1}{q} \leq 1 - \frac{n}{2}(\frac{1}{b} - \frac{1}{a}) \\ (\frac{1}{a}, \frac{1}{b}) \in \operatorname{int}(0 P_2 P_1) & \text{with } \frac{1}{q} \geq \frac{n}{2}(\frac{1}{a} - \frac{1}{b}) \end{cases}$$

Proof of Lemma 5

- As in the proof of Lemma 3, reduce to $2^j \le t - \tilde{t} < 2^{j+1}$, then to $j=0$, then reduce to F, G with temporal support in $O(1)$ interval.

- $\left(\frac{1}{a}, \frac{1}{b}\right) = O$ By the dispersive estimate

$$\left| \int_{1 \le t - \tilde{t} < 2} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)}\, dx\, dt\, d\tilde{x}\, d\tilde{t} \right| \lesssim \int_{1 \le t - \tilde{t} < 2} \frac{|F(\tilde{t},\tilde{x})|\,|G(t,x)|}{|t-\tilde{t}|^{n/2}}\, dx\, d\tilde{x}\, dt\, d\tilde{t} \quad |t-\tilde{t}|^{n/2} \sim 1$$

$$\lesssim \|F\|_{L^1_t L^1_x} \|G\|_{L^1_t L^1_x}$$

$$\lesssim \|F\|_{L^q_t L^1_x} \|G\|_{L^{q'}_t L^1_x}$$

$(\forall q \in [1,\infty])$

- $\left(\frac{1}{a}, \frac{1}{b}\right) = P_1$

$$\left| \int_{1 \le t - \tilde{t} < 2} K(t-\tilde{t}, x-\tilde{x}) F(\tilde{t}, \tilde{x}) \overline{G(t,x)}\, dx\, dt\, d\tilde{x}\, d\tilde{t} \right|$$

$$= \left| \int_{1 \le t - \tilde{t} < 2} \left\langle e^{i\tilde{t}\Delta} F(\tilde{t}, \cdot),\ e^{it\Delta} G(t,\cdot) \right\rangle_{L^2} d\tilde{t}\, dt \right|$$

$$\le \int_{1 \le t - \tilde{t} < 2} \|F(\tilde{t},\cdot)\|_{L^2_x} \|G(t,\cdot)\|_{L^2_x}\, d\tilde{t}\, dt$$

$$\le \|F\|_{L^1_t L^2_x} \|G\|_{L^1_t L^2_x} \lesssim \|F\|_{L^q_t L^2_x} \|G\|_{L^{q'}_t L^2_x}$$

$(\forall q \in [1,\infty])$

• $\underline{(\frac{1}{a},\frac{1}{b})=P_3}$ GOAL:

$$\left|\int_{1\le t-\tilde t<2} K(t-\tilde t,x-\tilde x)F(\tilde t,\tilde x)\overline{G(t,x)}\,dx\,dt\,d\tilde x\,d\tilde t\right| \lesssim \|F\|_{L^q_t L^{\frac{2n}{n+2}}_x}\|G\|_{L^{q'}_t L^2_x}$$

As in the proof of Lemma 3

$$\left|\int_{1\le t-\tilde t<2} K(t-\tilde t,x-\tilde x)F(\tilde t,\tilde x)\overline{G(t,x)}\,dx\,dt\,d\tilde x\,d\tilde t\right|$$

$$=\left|\int\left(\int e^{i(x-\tilde x)\cdot\xi-(t-\tilde t)|\xi|^2}H^{(t)}(\tilde t,\tilde x)\,d\xi\,d\tilde t\,d\tilde x\right)\overline{G(t,x)}\,dx\,dt\right|$$

$$\left(H^{(t)}=F\mathbb{1}_{(t-2,t-1]}\right)$$

$$=\left|\iiint\int e^{-i(\tilde x\cdot\xi-\tilde t|\xi|^2)}H^{(t)}(\tilde t,\tilde x)\,d\tilde t\,d\tilde x\int e^{i(x\cdot\xi-t|\xi|^2)}\overline{G(t,x)}\,dx\,d\xi\,dt\right|$$

$$\le \sup_t N_1(t)^{\frac12}\int N_2(t)^{\frac12}\,dt$$

$$N_1(t):=\int\left|\boxed{\overbrace{\int e^{-i(\tilde x\cdot\xi-\tilde t|\xi|^2)}H^{(t)}(\tilde t,\tilde x)\,d\tilde t\,d\tilde x}^{\widehat{U^*H^{(t)}}(\xi)}}\right|^2 d\xi$$

$$N_2(t):=\int\left|\int e^{-i(x\cdot\xi-t|\xi|^2)}G(t,x)\,dx\right|^2 d\xi$$

Now $N_2(t)\underset{\text{Plancherel}}{\simeq}\int|G(t,x)|^2dx$, so $\int N_2(t)^{\frac12}dt\simeq\|G\|_{L^1_tL^2_x}$

$$\lesssim\|G\|_{L^{q'}_tL^2_x}$$

$(\forall q\in[1,\infty])$

Also $N_1(t)\simeq\|U^*H^{(t)}\|_2^2\underset{\text{Keel-Tao}}{\lesssim}\|H^{(t)}\|^2_{L^2_tL^{\frac{2n}{n+2}}_x}$

$$\lesssim\|F\|^2_{L^2_tL^{\frac{2n}{n+2}}}\lesssim\|F\|^2_{L^q_tL^{\frac{2n}{n+2}}_x}$$

$(\forall q\in[2,\infty])$ ∎

Use Lemma 5 to get $T: \begin{array}{l} A_0 \times B_0 \to C_0 \\ A_0 \times B_1 \to C_1 \\ A_1 \times B_0 \to C_1 \end{array}$

$$A_j = L^q_t L^{a_j'}_x \qquad j=0,1$$
$$B_j = L^{q'}_t L^{a_j'}_x \qquad j=0,1$$
$$C_0 = \ell^\infty_{\beta(a_0,a_0)}$$
$$C_1 = \ell^\infty_{\beta(a_0,a_1)}$$

$$\frac{1}{a_0} = \frac{1}{r} + \varepsilon \qquad \left(\varepsilon > 0 : \text{ sufficiently small} \right)$$
$$\frac{1}{a_1} = \frac{1}{r} - 2\varepsilon$$

Apply Lemma 4 (with $p_0 = q$, $p_1 = q'$, $\theta_0 = 1/3 = \theta_1$), noting that

- $(C_0, C_1)_{2/3,1} = \ell^1_0$
- $(A_0, A_1)_{1/3,q} = (L^q_t L^{a_0'}_x, L^q_t L^{a_1'}_x)_{\frac{1}{3},q} = L^q_t L^{r',q}_x$ ← Lions-Peetre formula
- $(B_0, B_1)_{1/3,q'} = (L^{q'}_t L^{a_0'}_x, L^{q'}_t L^{a_1'}_x)_{1/3,q'} = L^{q'}_t L^{r',q'}_x$

So we obtain

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \lesssim \|F\|_{L^q_t L^{r',q}_x} \|G\|_{L^{q'}_t L^{r',q'}_x}$$
$$\lesssim \|F\|_{L^q_t L^{r'}_x} \|G\|_{L^q_t L^{r'}_x}$$

For the Lorentz space embedding we need $q \geq r'$

and $q' \geq r'$ i.e. $q \leq r$

i.e. $q \in [r', r] = \left[\frac{2n}{n+2}, \frac{2n}{n-2}\right]$. ∎