An introduction to Strichartz estimates I

Notation

- $A \lesssim B$ (or $B \gtrsim A$) means $A \leq CB$ where C is a constant.
  $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.
- $A \simeq B$ means $A = CB$ where C is a constant.

Schrödinger $e^{it\Delta} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \hat{f}(\xi)\, d\xi$

Notation $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$

Plancherel's theorem: $\|\hat{f}\|_2 \simeq \|f\|_2$.

Note $u(t,x) = e^{it\Delta} f(x)$ then $\begin{cases} i\partial_t u = -\Delta u \\ u(0) = f \end{cases}$

Formally
$$\left.\begin{aligned} \widehat{i\partial_t u}(t,\xi) &= i\partial_t \hat{u}(t,\xi) \\ -\widehat{\Delta u}(t,\xi) &= |\xi|^2 \hat{u}(t,\xi) \end{aligned}\right\}$$
so $\hat{u}(t,\xi) = A_\xi e^{-it|\xi|^2}$

Since $\hat{u}(0,\xi) = \hat{f}(\xi)$

we get

$\hat{u}(t,\xi) = \hat{f}(\xi) e^{-it|\xi|^2}$

Now use Fourier inversion.

Conservation of energy For each fixed $t \in \mathbb{R}$

$$\widehat{e^{it\Delta} f}(\xi) = e^{-it|\xi|^2} \hat{f}(\xi)$$

So Plancherel's theorem implies $\|e^{it\Delta} f\|_{L^2_x(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$

Notation • $\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^n)} \simeq \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi\right)^{\frac{1}{2}}$

"order s derivative"

Here $\widehat{(-\Delta)^{s/2} f}(\xi) = |\xi|^s \hat{f}(\xi)$.

$\dot{H}^s(\mathbb{R}^n)$ is the homogeneous Sobolev space order s.

Homogeneous Strichartz estimates

Set $s = \frac{n}{2} - \frac{n}{r} - \frac{2}{q}$ and consider

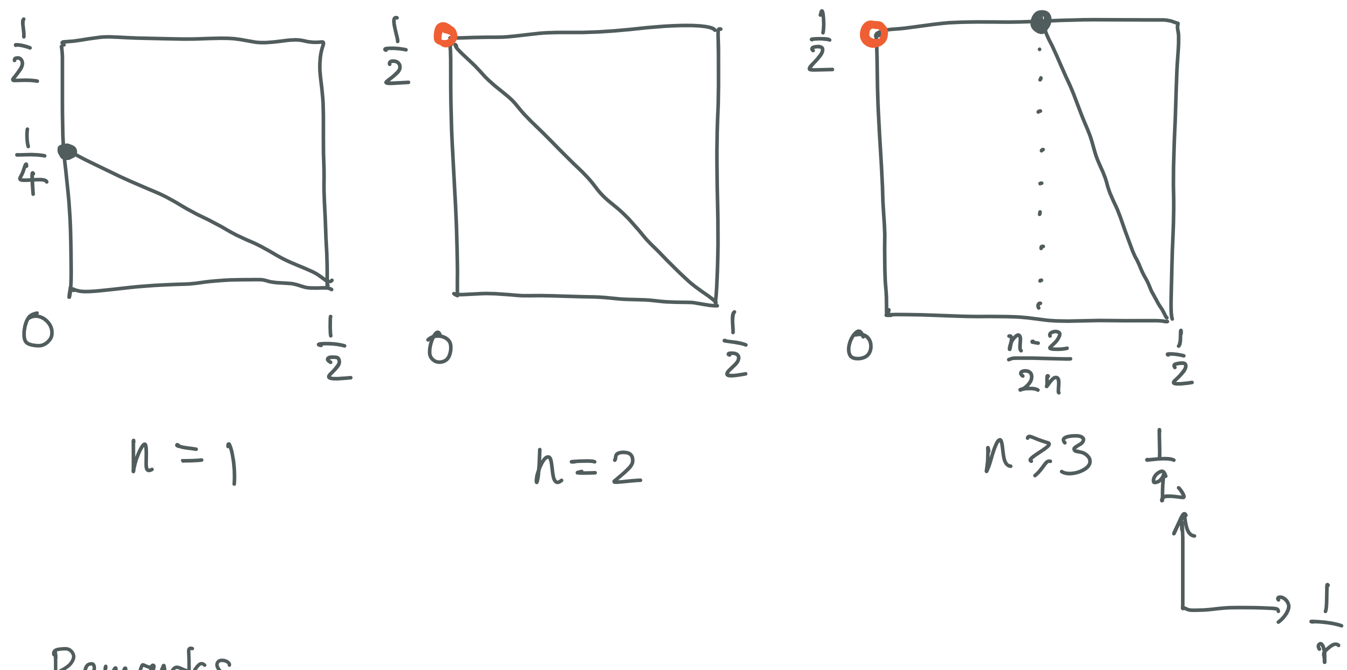
$$\|e^{it\Delta} f\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)} \qquad (H(q,r))$$

Notation $\|F\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |F(t,x)|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}$

Note $H(q,r)$ fails if $s \neq \frac{n}{2} - \frac{n}{r} - \frac{2}{q}$ (Rescaling argument)

Theorem 1 Let $n \geq 1$, $q, r \in [2,\infty]$, $\frac{1}{q} \leq \frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$, $s = \frac{n}{2} - \frac{n}{r} - \frac{2}{q}$.

- If $r < \infty$, then $H(q,r)$ holds.
- Let $r = \infty$.
  - If $n = 1$, $q \in [4,\infty)$ then $H(q,r)$ holds.
  - If $n \geq 2$, $q \in (2,\infty)$ then $H(q,r)$ holds.



Remarks

- $H(2, \frac{2n}{n-2})$ for $n \geq 3$ was proved by Keel–Tao (Amer. J. Math 1998) ~ see Keel-Tao for literature on "non-endpoint" cases.
- Failure of $H(2,\infty)$ when $n=2$ proved by Montgomery-Smith (Duke Math. J. 1998).
- Failure of $H(2,\infty)$ when $n \geq 3$ proved by Guo-Li-Nakanishi-Yan (J. Differential Equations 2018).
- Setting $Eg(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} g(\xi)\, d\xi$ we see that, when $s=0$, by Plancherel's theorem $H(\frac{2(n+2)}{n}, \frac{2(n+2)}{n})$ is equivalent to

$$\|Eg\|_{L^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})} \lesssim \|g\|_{L^2} \qquad (*)$$

The analogue for the sphere

$$\left\| \int_{S^{d-1}} e^{ix\cdot\theta} g(\theta)\, d\sigma(\theta) \right\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}^d)} \lesssim \|g\|_{L^2(d\sigma)}$$

is known as the Stein-Tomas estimate (established shortly before Strichartz's work).

Also it is known that gaussians are maximisers for $(*)$ when $n=1,2$ and conjectured to be true for all $n \geq 3$. See, e.g., Foschi (J. Eur. Math. Soc. 2007).

Preparation for proof of Theorem 1 $(r<\infty)$

Notation • $Uf(t,x) = e^{it\Delta} f(x)$

• Fix non-negative, radial $\chi \in C_c^\infty$ with

$$\operatorname{supp} \chi \subseteq \{ \xi \in \mathbb{R}^n : |\xi| \in [1/2, 2] \}$$

$$\sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi) = 1 \quad (\forall \xi \neq 0)$$

"smooth partition of unity"

• $\widehat{P_k f}(\xi) = \chi_k(\xi)\hat{f}(\xi)$, $\chi_k(\xi) = \chi(2^{-k}\xi)$

In particular, $f = \sum_{k\in\mathbb{Z}} P_k f$.

• $U_0 := U P_0$

Lemma 1 (Frequency localisation)

$$\|U_0 f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2} \implies H(q,r)$$

Proof

- Rescaling: For any $k \in \mathbb{Z}$, setting $\hat{f}_k(\xi) := 2^{k\frac{n}{2}} \hat{f}(2^k \xi)$

we have

$$UP_k f(t,x) \simeq \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \chi(\underbrace{2^{-k}\xi}_{=\eta}) \hat{f}(\xi)\, d\xi$$

$$= 2^{k\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(2^k x\cdot\eta - 2^{2k}t|\eta|^2)} \chi(\eta) \hat{f}_k(\eta)\, d\eta$$

$$= 2^{k\frac{n}{2}} UP_0 f_k(2^{2k}t, 2^k x)$$

So

$$\|UP_k f\|_{L^q_t L^r_x} = \|UP_k \tilde{P}_k f\|_{L^q_t L^r_x}$$

$$= \frac{2^{\frac{kn}{2}}}{2^{\frac{2k}{q}} 2^{\frac{nk}{r}}} \|UP_0 \tilde{P}_0 f_k\|_{L^q_t L^r_x}$$

$$\lesssim 2^{ks} \|\tilde{P}_0 f_k\|_{L^2}$$

Need $P_k f = P_k \tilde{P}_k f$ so we define $\tilde{P}_k$ as in the definition of P_k but with $\tilde{\chi}(\xi) = 1$ on the support of χ.

Hence

$$\|Uf\|_{L^q_t L^r_x} = \Big\| \sum_{k\in\mathbb{Z}} P_k Uf \Big\|_{L^q_t L^r_x}$$

Littlewood-Paley theory (ok since $r<\infty$) $\rightarrow$

$$\lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |P_k Uf|^2\Big)^{\frac{1}{2}} \Big\|_{L^q_t L^r_x}$$

$q, r \geq 2 \rightarrow$

$$\leq \Big(\sum_{k\in\mathbb{Z}} \|P_k Uf\|^2_{L^q_t L^r_x}\Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{2ks} \|\tilde{P}_0 f_k\|^2_{L^2}\Big)^{\frac{1}{2}}$$

But

$$\sum_{k\in\mathbb{Z}} 2^{2ks}\int \tilde{\chi}(\xi)^2\, 2^{kn}\, |\hat{f}(2^k\xi)|^2\, d\xi$$

$$= \sum_{k\in\mathbb{Z}} 2^{2ks}\int \tilde{\chi}(2^{-k}\xi)^2\, |\hat{f}(\xi)|^2\, d\xi$$

$|\xi|\sim 2^k$ on supp $\tilde{\chi}$

$$\sim \int \sum_{k\in\mathbb{Z}} \tilde{\chi}(2^{-k}\xi)^2\, |\hat{f}(\xi)|^2\, |\xi|^{2s}\, d\xi$$

$\lesssim 1$

$$\lesssim \|f\|^2_{\dot{H}^s}.$$

Duality $\|U_0 f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2} \Leftrightarrow \|U_0^* F\|_{L^2} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}$

$$\Leftrightarrow \|U_0 U_0^* F\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}$$

$$\Leftrightarrow |\langle U_0 U_0^* F, G\rangle| \lesssim \|F\|_{L^{q'}_t L^{r'}_x} \|G\|_{L^{q'}_t L^{r'}_x}$$

Now

$$\langle U_0 f, F\rangle \simeq \int \hat{f}(\xi) \int e^{i(x\cdot\xi - t|\xi|^2)} \chi(\xi)\, \overline{F(t,x)}\, dt\, dx\, d\xi$$

$$\langle f, U_0^* F\rangle \simeq \langle \hat{f}, \widehat{U_0^* F}\rangle \quad \text{implies}$$

$$\widehat{U_0^* F}(\xi) \simeq \int e^{-i(x\cdot\xi - t|\xi|^2)} \chi(\xi)\, F(t,x)\, dt\, dx$$

$$= \chi(\xi)\, \hat{F}(-|\xi|^2, \xi)$$

(Restriction of Fourier transform)

and

$$\mathcal{U}_0\mathcal{U}_0^* F(t,x) \simeq \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}\chi(\xi)^2 \hat{F}(-|\xi|^2,\xi)\,d\xi$$

$$= \int_{\mathbb{R}^{n+1}} F(\tilde{t},\tilde{x}) \int_{\mathbb{R}^n} \chi(\xi)^2 e^{i((x-\tilde{x})\cdot\xi - (t-\tilde{t})|\xi|^2)}\,d\xi\,d\tilde{t}\,d\tilde{x}$$

$$= F * K(t,x), \quad \text{where}$$

$$K(t,x) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}\chi(\xi)^2\,d\xi$$

Note Setting $K_t(x) = K(t,x)$, we have $\hat{K_t}(\xi) = e^{-it|\xi|^2}\chi(\xi)^2$.

Lemma 2 (Oscillatory integral estimate / dispersive estimate)

For any $n \geq 1$, $\sup_{x\in\mathbb{R}^n} |K(t,x)| \lesssim \dfrac{1}{(1+|t|)^{n/2}}$ $(\forall t \in \mathbb{R})$

Proof when $n=1$ and $\chi \rightsquigarrow 1_{[\frac{1}{2},2]}$

(Argument based on the standard proof of van der Corput's lemma)

Let us prove $\displaystyle\sup_{x\in\mathbb{R}} \left|\int_{1/2}^{2} e^{i\phi(\xi)}d\xi\right| \lesssim \frac{1}{(1+|t|)^{1/2}}$

where $\phi(\xi) = x\xi - t\xi^2$.

- If $|t| \leq 1$, estimate trivially $\displaystyle\left|\int_{1/2}^{2} e^{i\phi(\xi)}d\xi\right| \lesssim 1 \lesssim \frac{1}{(1+|t|)^{1/2}}$.

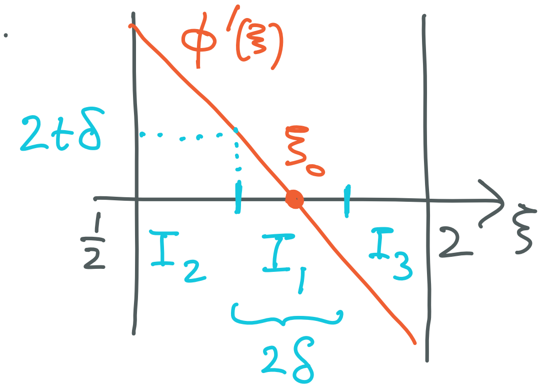
- Now suppose $|t| \geq 1$, and (without loss of generality) $t \geq 1$.

We have $\phi'(\xi) = x - 2t\xi$, $\phi''(\xi) = -2t$.

Let $\xi_0 = \dfrac{x}{2t}$, so that $\phi'(\xi_0) = 0$.

Write $I_1 = (\xi_0 - \delta, \xi_0 + \delta)$

($0 < \delta << 1$ is chosen later)



On I_1, estimate trivially

$$\left| \int_{I_1} e^{i\phi(\xi)} d\xi \right| \leq \int_{I_1} 1 \, d\xi = 2\delta .$$

On I_2:
$$\int_{I_2} e^{i\phi(\xi)} d\xi = \int_{1/2}^{\xi_0 - \delta} \left(e^{i\phi(\xi)}\right)' \frac{1}{i\phi'(\xi)} d\xi$$

$$= \left[\frac{e^{i\phi(\xi)}}{i\phi'(\xi)} \right]_{1/2}^{\xi_0 - \delta} - \frac{1}{i} \int_{1/2}^{\xi_0 - \delta} e^{i\phi(\xi)} \left(\frac{1}{\phi'(\xi)} \right)' d\xi$$

so
$$\left| \int_{I_2} e^{i\phi(\xi)} d\xi \right| \leq \frac{1}{t\delta} + \int_{1/2}^{\xi_0 - \delta} \left(\frac{1}{\phi'(\xi)} \right)' d\xi$$

$$= \frac{1}{t\delta} + \frac{1}{\phi'(\xi_0 - \delta)} - \frac{1}{\phi'(\frac{1}{2})} \leq \frac{3}{2t\delta}$$

Similarly
$$\left| \int_{I_3} e^{i\phi(\xi)} d\xi \right| \leq \frac{3}{2t\delta} .$$

So

$$\left|\int_{1/2}^{2} e^{i\phi(\xi)}\,d\xi\right| \leq 2\delta + \frac{3}{t\delta}$$ and optimising in δ (i.e. choose $\delta = \frac{1}{t^{1/2}}$) gives $\left|\int_{1/2}^{2} e^{i\phi(\xi)}d\xi\right| \lesssim \frac{1}{t^{1/2}}$ ∎

Remarks (1) Chapters 8–9 of Stein's book (Harmonic Analysis, Princeton Uni. Press 1993) is an excellent resource for the theory of oscillatory integrals.

(2). Formally, if $\chi \equiv 1$ then $K(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-z|\xi|^2} d\xi$ $(z = it)$

If $z \in (0,\infty)$, then $\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-z|\xi|^2} d\xi \simeq \dfrac{e^{-\frac{|x|^2}{4z}}}{z^{n/2}}$

and this identity can be made sense of when $z = it$ $(t \neq 0)$ by using distributions.

Goal $U_0 f(t,x) = U P_0 f(t,x)$

$$\boxed{|\langle U_0 U_0^* F, G\rangle| \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}} \quad (H_0(q,r))}$$

"Non-endpoint cases" i.e. (q,r) as in Theorem 1 with $(q,r) \neq (2, \frac{2n}{n-2})$ for $n \geq 3$, and $r < \infty$.

Fix $t, \hat{t} \in \mathbb{R}$, write $F_{\tilde{t}}(\hat{x}) = F(\hat{t}, \hat{x})$.

Case $r = 2$

$$\left| \int K(t-\hat{t}, x-\hat{x}) F(\hat{t}, \hat{x}) \overline{G(t,x)} \, dx \, d\hat{x} \right|$$

$$\leq \|G_t\|_2 \left(\int \left| \int K(t-\hat{t}, x-\tilde{x}) F(\hat{t}, \tilde{x}) \, d\tilde{x} \right|^2 dx \right)^{1/2}$$

$$= \|G_t\|_2 \left(\int \left| \int K_{t-\hat{t}}(x-\tilde{x}) F_{\tilde{t}}(\tilde{x}) \, d\tilde{x} \right|^2 dx \right)^{1/2}$$

$$= \|G_t\|_2 \, \| K_{t-\tilde{t}} * F_{\tilde{t}} \|_2$$

$$\simeq \|G_t\|_2 \, \| \underbrace{\widehat{K_{t-\tilde{t}}}}_{\lesssim 1} \widehat{F_{\tilde{t}}} \|_2 \lesssim \|F_{\tilde{t}}\|_2 \|G_t\|_2$$

Case $r = \infty$ By Lemma 2

$$\left| \int K(t-\hat{t}, x-\hat{x}) F(\hat{t}, \hat{x}) \overline{G(t,x)} \, dx \, d\hat{x} \right| \lesssim \frac{\|F_{\tilde{t}}\|_1 \|G_t\|_1}{(1+|t-\hat{t}|)^{n/2}}$$

By interpolation

$$\left| \int K(t-\hat{t}, x-\hat{x}) F(\hat{t}, \hat{x}) \overline{G(t,x)} \, dx \, d\hat{x} \right| \lesssim \frac{\|F_{\tilde{t}}\|_{r'} \|G_t\|_{r'}}{(1+|t-\hat{t}|)^{\frac{n}{2}(1-\frac{2}{r})}}$$

So

$$|\langle U_0 U_0^* F, G\rangle| \lesssim \int_{\mathbb{R}^2} \frac{\|F_{\tilde t}\|_{r'} \|G_t\|_{r'}}{(1+|t-\tilde t|)^{\frac n2(1-\frac 2r)}} \, dt\, d\tilde t$$

$\frac n2(1-\frac2r) = \frac 2q$ when $\frac1q = \frac n2\left(\frac12-\frac1r\right)$

Young's convolution

$$\lesssim \big\| \|F_{\tilde t}\|_{r'}\big\|_{q'} \big\| \|G_t\|_{r'}\big\|_{q'} \left\| \frac{1}{(1+|\cdot|)^{\frac n2(1-\frac2r)}}\right\|_{\frac q2}$$

$$\begin{cases} q\in[2,\infty] \\ \frac1q < \frac n2\left(\frac12-\frac1r\right) \end{cases}$$

Hardy-Littlewood-Sobolev inequality

$$\lesssim \big\| \|F_{\tilde t}\|_{r'}\big\|_{q'} \big\| \|G_t\|_{r'}\big\|_{q'}$$

$q\in(2,\infty]$

$\frac1q = \frac n2\left(\frac12-\frac1r\right)$

In conclusion, we have proved $H_0(q,r)$ whenever

$$n\geq 1,\ q\in[2,\infty],\ r\in[2,\infty],\ \frac1q\leq\frac n2\left(\frac12-\frac1r\right),$$
$$(q,r)\neq\left(2,\frac{2n}{n-2}\right)\ (n\geq 2)$$

Next we handle the "endpoint case":

Theorem 2 (Keel-Tao, Amer. J. Math 1998)

$H_0\left(2,\frac{2n}{n-2}\right)$ holds for all $n\geq 3$.